Putnam Club
September 25, 2013
Combinatorics

1. Each of the faces of the cube is colored a different color. How many of the colorings are distinct?

2. Find the sum
$$\sum_{k=1}^{n} \binom{n}{k} k^3.$$
(Consider the problem of selecting a committee, and a chairman, vice-chairman, and a secretary in this committee.)

3. How many ways are there to choose n objects from $3n+1$ objects, assuming that of these $3n+1$, n objects are indistinguishable, and the rest are all distinct?

4. How many subsets of $\{1, \ldots, n\}$ have no two successive numbers?

5. Can we arrange the numbers $1, 2, \ldots, 9$ along a circle so that the sum of two neighbors is never divisible by 3, 5, or 7?

6. Consider a circular row of n seats; a child seats on each. Each child can move by at most one seat. Find the number of ways in which they can rearrange.

7. Is there a subset $A \subset \{1, \ldots, 3000\}$ with 2000 elements such that if $x \in A$, then $2x \notin A$.

8. Does a polyhedron exist with an odd number of faces, each face having an odd number of edges?

9. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a minimal element. Let $F(n, r)$ denote the mean of these smallest numbers; show that
$$F(n, r) = \frac{n+1}{r+1}.$$