Fall 2017

Equations with functions as unknowns

Thursday, September 27th, 2017

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1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 - y^2) = (x - y)\left(f(x) + f(y)\right).$$

2. Find all functions $f : \mathbb{R} - \{1\} \to \mathbb{R}$, continuous at 0, that satisfy

$$f(x) = f\left(\frac{x}{1-x}\right), \quad x \in \mathbb{R} - \{1\}.$$

3. Find all functions $f : [0, 1] \to \mathbb{R}$ satisfying the following conditions

   - $[f(x)]\sin^2 x + [x]\cos f(x)\cos x = f(x)$
   - $f(f(x)) = f(x)$.

   Here $[x]$ means the fractional part of x.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, z, we have

$$[f(x) + 1]\,[f(y) + f(z)] = f(xy + z) + f(xz - y).$$

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(f(x) + y\right) = f\left(f(x) - y\right) + 4f(x)y, \text{ for any } x, y \in \mathbb{R}.$$

6. Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined by

$$a_1 := c, \text{ and } a_{n+1} = a_n^2 + a_n + c^3, n \in \mathbb{N}.$$

   Find all values of c for which there exist some integers $k \geq 1$ and $m \geq 2$ such that $a_k^2 + c^3$ is the m-th power of some positive integer. Solution: Notice that

$$a_{n+1}^2 + c^3 = (a_n^2 + a_n + c^3)^2 + c^3 = (a_n^2 + c^3)(a_n^2 + 2a_n + 1 + c^3).$$

   Claim: $a_n^2 + c^3$ and $a_n^2 + 2a_n + 1 + c^3$ are coprime.
   Proof of the Claim: Firts, we prove that $4c^3 + 1$ is coprime with $2a_n + 1$, for every $n \geq 1$. Let $n = 1$, and p be a prime divisor of $4c^3 + 1$ and $2a_1 + 1 = 2c + 1$. Then p divides $2(4c^3 + 1) = (2c + 1)(4c^2 - 2c + 1) + 1$, hence p divides 1, which is a contradiction. Assume now that $(4c^3 + 1, 2a_n + 1) = 1$ for some $n \geq 1$, hence the prime p divides $4c^3 + 1$ and $2a_{n+1} + 1$. Then p divides $4a_{n+1} + 2 = (2a_n + 1)^2 + 4c^3 + 1$, which again gives a contradiction.

   Assume that for some $n \geq 1$ the number

$$a_{n+1}^2 + c^3 = (a_n^2 + a_n + c^3)^2 + c^3 = (a_n^2 + c^3)(a_n^2 + 2a_n + 1 + c^3).$$

is a power. Since $a_n^2 + c^3$ and $a_n^2 + 2a_n + 1 + c^3$ are coprime, then $a_n^2 + c^3$ is a power as well.

The same argument can be further applied given that

$$a_1^3 + c^3 = c^2 + c^3 = c^2(c+1)$$

is a power. If $a^2(a+1) = t^m$ with odd $m \geq 3$, then $a = t_1^m$ and $a + 1 = t_2^m$, which is impossible. If $a^2(a+1) = t^{2m_1}$ with $m_1 \geq 2$ then $a = t_1^{m_1}$ and $a + 1 = t_2^{m_2}$, which is again impossible.

Therefore $a^2(a+1) = t^{2m_1} = t^2$, which implies that $a = s^2 - 1$ for $s \geq 2$, and $s \in \mathbb{N}$.

7. (Putnam) Find all functions f from the interval $(1, \infty)$ to $(1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

8. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)},$$

for all $x > 0$.