Fall 2017

Analysis problems and a mix of random questions (mainly from probability theory)

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1. Let $f : [0, \infty) \to \mathbb{R}$ be a twice-differentiable function satisfying $f(0) \geq 0$ and $f'(x) > f(x)$ for all $x > 0$. Prove that $f(x) > 0$ for all $x > 0$.

2. Let f be a real-valued continuous nonnegative function on $[0, 1]$ such that

$$f(t)^2 \leq 1 + 2\int_0^t f(s)\, ds, \quad \text{for all } t \in [0, 1]\,.$$

   Show that $f(t) \leq 1 + t$ for every $t \in [0, 1]$.

3. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) > 0$ and $f'(x) = f(f(x))$ for every $x \in \mathbb{R}$?

4. Determine all nth-degree polynomials $P(x)$, with real zeros, for which the equality

$$\sum_{i=1}^{n} \frac{1}{P(x) - x_i} = \frac{n^2}{xP'(x)}$$

   holds for all nonzero real numbers x for which $P'(x) \neq 0$, where x_i, $i = 1, 2, \cdots, n$, are the zeros of $P(x)$.

5. Let C be the class of all real-valued continuously differentiable functions f on the interval $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Determine

$$u = \inf_{f \in C} \int_0^1 |f'(x) - f(x)|\, dx.$$

6. Prove that

$$\begin{bmatrix} 2k \\ k \end{bmatrix} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (2 \sin \theta)^{2k}\, d\theta.$$

7. Show that for any positive integer n, the number

$$S_n = \begin{bmatrix} 2n+1 \\ 0 \end{bmatrix} \cdot 2^{2n} + \begin{bmatrix} 2n+1 \\ 2 \end{bmatrix} \cdot 2^{2n-2} \cdot 3 + \cdots + \begin{bmatrix} 2n+1 \\ 2n \end{bmatrix} \cdot 3^n$$

   is the sum of two consecutive perfect squares.

8. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence?

9. What is the probability that three points selected at random on a circle lie on a semicircle?

10. Let $n \geq 4$ be given, and suppose that the points $P_1, P_2, \ldots . P_n$ are randomly chosen on a circle. Consider the convex n-gon whose vertices are these points. what is the probability that at least one of the vertex angles of this polygon is acute?