Some matrix problems from Virginia Tech Math competition.
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Almost every year, the competition includes a problem on matrices (or more general linear algebra). Here are some examples.

1. Let $\mathbb{C}$ denote the complex numbers and let $M_3(\mathbb{C})$ denote the 3-by-3 matrices with entries in $\mathbb{C}$. Suppose $A, B \in M_3(\mathbb{C})$, $B \neq 0$, and $AB = 0$ (where 0 denotes the 3-by-3 matrix with all entries zero). Prove that there exists

$$0 \neq D \in M_3(\mathbb{C})$$

such that

$$AD = DA = 0.$$

2. Determine all invertible 2-by-2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A.

3. (I really hate this one) Let I denote the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}, N = \begin{pmatrix} I & B \\ A & C \end{pmatrix},$$

where A, B, C are arbitrary 2×2 matrices with entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

4. Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ijth entry of A' is the jith entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form Au where u is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'v$ where v is a 5×1 matrix with real entries.

5. Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that

$$\det(AX + BY) \neq 0.$$

Prove that

$$\det(A^2 + B^2) \neq 0$$

(det indicates the determinant).

6. Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1, 1\}$ or $\{1, -1\}$ or $\{-1, -1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in S ($A = B$ is allowed).