GENERATING FUNCTIONS-PUTNAM SEMINAR 2016

1. (M) The set of natural numbers is partitioned into finitely many arithmetic progression $\{a_i + dr_i\}$, $1 \le i \le n$. Prove that :
   - $\displaystyle\sum_{i=1}^{n} \frac{1}{r_i} = 1$
   - $\displaystyle\sum_{i=1}^{n} \frac{a_i}{r_i} = \frac{n-1}{2}$
   - There exist $i \neq j$ such that $r_i = r_j$
2. (H) Find all natural numbers n for which there exist two distinct sets of integers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ such that the sets $\{a_i + a_j\}$ and $\{b_i + b_j\}$, $1 \le i < j \le n$ coincide.
3. (M) How many polynomials with coefficients $0, 1, 2, 3$ are there such that $P(2) = n$, where n is a given positive integer?
4. (E) For $r \in \mathbb{N}^*$ let a_r denote the number of solutions to the equation
$$x_1 + x_2 + x_3 + x_4 \le r$$
where $3 \le x_1 \le 9$, $1 \le x_2 \le 10$, $x_3 \ge 2$ and $x_4 \ge 0$. Find the generating function for a_r and use it to find the value of a_{20}.
5. (E) Let X be the set of triplets (a, b, c) of nonnegative integers such that $a + b + c = 2008$. Let S be the sum of abc over all triplets in X. Prove that 1004 divides S.
6. (E) Let S_n be the number of triplets of nonnegative integers (a, b, c) such that $a+2b+3c = n$. Compute the sum $\displaystyle\sum_{n=0}^{\infty} \frac{S_n}{3^n}$.
7. (E) A deck of 32 cards has 2 different jokers each of which is numbered 0. There are 10 red cards numbered 1 through 10 and similarly for blue and green cards. One chooses a number of cards from the deck to form a hand. If a card in the hand is numbered k, then the value of the card is 2^k, and the value of the hand is sum of the values of the cards in hand. Determine the number of hands having the value 2004.
8. (M) Let n be a positive integer. Let $d(n)$ denote the number of partitions of n with distinct parts and let $o(n)$ equal the number of partitions of n with odd parts. Prove that $d(n) = o(n)$.
9. (M) Let n be a positive integer. Show that the number of partitions of n, where each part appears at least twice, is equal to the number of partitions of n into parts that are divisible by 2 or 3.
10. (M) Let n be a positive integer. Show that the number of partitions of n into parts which have at most one of each distinct even part equals the number of partitions of n in which each part can appear at most 3 times.
11. (H) Find all partitions with two classes of the set of nonnegative integers having the property that for all nonnegative integers n the equation $x+y = n$, $x < y$ has as many solutions (x, y) in $A \times A$ as in $B \times B$.
12. (M) How many ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 20\}$ can we find such that each element of A is larger than $|B|$ and each element of B is larger than $|A|$?
13. (M) Prove that $\displaystyle\sum_{n \ge 0} \binom{n}{k} X^n = \frac{X^k}{(1-X)^{k+1}}$
14. (M) Prove that $\displaystyle\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} 2^k$