**Number theory.**
**10/09/13**

Number theory is a common topic.

**1.** Among five integers, there are always three with sum divisible by 3.

**2.** Among $n+1$ positive integers no greater than $2n$, there are two that are coprime.

**3.** If $p$ and $p^2+2$ are primes, then $p^3+2$ is also a prime.

**4.** (VT 2006, #3) Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

**5.** (VT 2011, #4). Let $m, n$ be positive integers and let $[a]$ denote the residue class $\mod mn$ of the integer $a$ (thus

$$\{[r] | r \text{ is an integer}\}$$

has exactly $mn$ elements). Suppose the set

$$\{[ar] | r \text{ is an integer}\}$$

has exactly $m$ elements. Prove that there is a positive integer $q$ such that $q$ is prime to $mn$ and $[nq] = [a]$.

**6.** (VT 2012, #4). Define $f(n)$ for $n$ a positive integer by

$$f(1) = 3 \text{ and } f(n+1) = 3^{f(n)}.$$

What are the last two digits of $f(2012)$?

**7.** (Putnam 2003, B3). Show that

$$\prod_{i=1}^{n} \operatorname{lcm}(1, 2, 3, \ldots, [n/i]) = n!$$

**8.** (Putnam 2000, A6). $p(x)$ is a polynomial with integer coefficients. A sequence $x_0, x_1, x_2, \ldots$ is defined by

$$x_0 = 0, \quad x_{n+1} = p(x_n).$$

Prove that if $x_n = 0$ for some $n > 0$, then $x_1 = 0$ or $x_2 = 0$.