

POLYNOMIALS (10/11/17)

EASIER PROBLEMS

1. The polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ has integer coefficients and n distinct integer roots. Suppose that all of its roots are coprime. Show that a_n and a_{n-1} are coprime.
2. a , b , and c are the three roots of the polynomial $x^3 - 3x^2 + 1$. Find $a^3 + b^3 + c^3$.
3. $p(x)$ is a polynomial with integer coefficients such that $p(0) = 1$, $p(1) = 2$, $p(-1) = 1$. Prove that $p(x)$ has no integer roots.
4. (AIME 1989) Assume that x_1, x_2, \dots, x_7 are real numbers such that

$$\begin{aligned}x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123.\end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

PUTNAM PROBLEMS (NOT NECESSARILY HARDER)

5. (Putnam 2007) Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.
6. (Putnam 2009) Let $p(x)$ be a real polynomial that is nonnegative for all real x . Prove that for some k , there are real polynomials $f_1(x), \dots, f_k(x)$ such that $p(x) = \sum_{i=1}^k f_i(x)^2$.
7. (Putnam 2003) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

8. (Putnam 2010) Find all polynomials $P(x), Q(x)$ with real coefficients such that $P(x)Q(x+1) - P(x+1)Q(x) = 1$.
9. (Putnam 2008) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points

$$(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n)) \in \mathbb{R}^2$$

are the vertices of a regular n -gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater or equal to $n - 1$. (By the way, suppose n is odd and $\deg(f(x)) < n - 1$. Can you say anything special about the regular n -gon?)

10. (Putnam 2005) Let $p(z)$ be a polynomial of degree n all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z)$ have absolute value 1.

GOOD THINGS TO KNOW

- The Fundamental Theorem of Algebra.
- Expressions for coefficients in terms of roots (as elementary symmetric polynomials).
- Roots of real polynomials (must occur in conjugate pairs).
- Intermediate Value Theorem.
- Mean Value Theorem: For a real-valued function, the derivative must vanish between two zeros of the function.
- Divisibility: for an integer polynomial $p(x)$, $(a - b) \mid (p(a) - p(b))$.

HINTS

1. Think about how a_i 's are expressed in terms of roots.
2. Express $a^3 + b^3 + c^3$ in terms of the elementary symmetric polynomials.
3. What is $p(x)$ modulo 3?
4. The left-hand sides are values of a (low-degree!) polynomial.
5. Not much to say - straight divisibility question.
6. What does the factorization of $p(x)$ look like?
7. Consider the coefficients of y^k on the right-hand side as polynomials of x . How are they related to $a(x)$ and $b(x)$? (Or plug in particular values of y into both sides.)
8. Compare $P(x)(Q(x + 1) - Q(x))$ and $Q(x)(P(x + 1) - P(x))$.
9. What can you say about the vector-valued polynomial function
$$(f(x + 1) - f(x), g(x + 1) - g(x))$$

?- 10. What is special about the values of $g(z)$ if $|z| = 1$? Also, think about the Mean Value Theorem.