LINEAR ALGEBRA (MOSTLY DETERMINANTS)
(10/14/15)

1. Consider all possible 3×3 matrices whose entries are numbers 0, 1, or 2. (There are 3^9 of them.) Prove that the sum of their determinants is zero.

2. Let A and B be 3×3 matrices with integer entries. Suppose $A + xB$ is invertible and its inverse has an integer entries for $x = 0, 1, 2, 3, 4, 5, 6$. Prove that this holds for any integer x.

3. Let A be an $n \times n$ matrix whose all entries are odd integers. Prove that $\det(A)$ is divisible by 2^{n-1}.

4. Suppose A and B are $n \times n$ matrices such that $AB = A + B$. Show that $BA = A + B$.

5. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy? (Putnam 2008)

6. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example,

$$d_3 = \det \begin{pmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{pmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$. (Putnam 2009)

(Suggestion: what happens if you use e^k instead of $\cos(k)$?)

Properties of determinants

Let A be an $n \times n$ matrix. Its *determinant* is an expression that is a sum of terms of the form $a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}$ for all possible permutations σ; each term has a sign given by the *sign* of permutation σ.

Computing the determinant directly is usually too complicated, but there are certain properties that can help you:

- If you add (or subtract) a multiple of one row from another row, the determinant does not change;
- Scaling a row by a constant scales the determinant by the same constant;
- Swapping two rows (not necessarily adjacent) changes the sign of the determinant;
- If two rows are equal or proportional, the determinant is zero. If one of the rows contains only zeros, the determinant is zero.

The same properties holds if one works with columns instead of rows.

Finally, the determinant of an upper triangular matrix is the product of its diagonal entries.

What are determinants good for?

The determinant is zero if and only if rows of the matrix are linearly dependent (and also if and only if the columns are linearly dependent).

Also, $\det(A) \neq 0$ iff there is B such that $AB = I$ (right inverse) and iff there is B such that $BA = I$ (left inverse); if the inverses exist, they are equal.

There is an explicit (but complicated) formula for the inverse A^{-1}; it has some algebraic expression in the numerator and $\det(A)$ in the denominator.

$\det(AB) = \det(A)\det(B)$. If A is invertible, $\det(A^{-1}) = \det(A)^{-1}$.

What next?

Here are some other important keywords from linear algebra: trace, eigenvalues and eigenvectors, characteristic polynomial, the Cayley-Hamilton Theorem, Vandermonde determinant.