Functions and calculus.

11/01/17

1. Find all polynomials \( p(x) \) satisfying
   \[ p(x + 1) = p(x) + 2x + 1. \]

2. Find all functions \( f \) with the property that
   \[ f(x) = f(x/2) \]
   for all \( x \in \mathbb{R} \).

3. (VT 2007, #2). Given that
   \[ e^x = 1/0! + x/1! + x^2/2! + \cdots + x^n/n! + \ldots, \]
   find, in terms of \( e \), the exact values of
   \[ 1/1! + 2/3! + 3/5! + \cdots + n/(2n - 1)! + \ldots \]
   and
   \[ 1/3! + 2/5! + 3/7! + \cdots + n/(2n + 1)! + \ldots. \]

4. (VT 2008, #1). Find the maximum value of
   \[ xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x \]
   where \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \).

5. (VT 2011, #7). Let
   \[ P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1x + 1 \]
   be a polynomial where the \( a_i \) ( \( 1 \leq i \leq 97 \)) are real numbers. Prove that the equation \( P(x) = 0 \) has at least one complex root (i.e., a root of the form \( a + bi \) with \( a, b \) real numbers and \( b \neq 0 \)).

6. (Putnam 2009, A1). Let \( f \) be a real-valued function on the plane such that for every square \( ABCD \) in the plane,
   \[ f(A) + f(B) + f(C) + f(D) = 0. \]
   Does it follow that \( f(P) = 0 \) for all points \( P \) in the plane?

7. (Putnam 2010, A2). Find all differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) such that
   \[ f'(x) = \frac{f(x + n) - f(x)}{n} \]
   for all real numbers \( x \) and all positive integers \( n \).

8. (Putnam 2008, B5). Find all continuously differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) such that for every rational number \( q \), the number \( f(q) \) is rational and has the same denominator as \( q \). (The denominator of a rational number \( q \) is the unique positive integer \( b \) such that \( q = a/b \) for some integer \( a \) with \( \text{gcd}(a, b) = 1 \).)
   (Note: \( \text{gcd} \) means greatest common divisor.)
Hints:

1. What is $\deg(p(x + 1) - p(x))$?

2. Once you are allowed to multiply or divide $x$ by 2, is there some ‘normal form’ that you can bring it to?

3. What is the relation between the Taylor power series of $f(x)$ and that of $f(x) + f(-x)$? that of $f'(x)$?

4. Note the symmetry of the expression, and try maximizing it one variable at a time.

5. Use the fact that the derivative must vanish at least once between the roots of the polynomial.

6. The trick is to combine several instances of the relation in a non-trivial way.

7. What can you say about the derivative of $f$?

8. Use the linear approximation of $f$ near a rational point (say, 0) to determine $f$’s derivative at that point.