Pigeonhole principle.
11/13/13

Warm-up: Sixteen distinct integers from 1 to 30 are written on the board. Prove that you can always find two of them such that (a) they are coprime (b) one of them divides the other (c) they are congruent modulo 3.

1. Show that there exists a Fibonacci number that is divisible by 2013. (The Fibonacci sequence is defined by $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.)

2. Ten integers between 1 and 100 are written on the board. Prove that you can always find two disjoint subsets with the same sum of elements.

3. Given nine points inside a unit square (i.e., a square with side length 1), prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.

4. A sequence of 2013 numbers is written on the board. Each number is an integer between 1 and 30. Show that there exists a continuous (nonempty) block within this sequence such that the product of all numbers in this block is a perfect square.

5. 10 dinner guests seat at a round table. They realize that some of them are not in their assigned seats. Prove that there is a rotation of the table such that at least two guests are in their assigned seats.

6. (Putnam'95, B1) For a partition π of the set $\{1, \ldots, 9\}$, let $\pi(x)$ be the number of the elements in the part containing x. Prove that for any two partitions π and π', there are two distinct elements x and y such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. (A *partition* is a collection of disjoint sets (*parts*) whose union is $\{1, \ldots, 9\}$.)

Time permitting, I'd also like to look at leftovers from the last meeting:

7. (Putnam 2006, A2). Alice and Bob play a game in which they take turns removing stones from a heap that initially had n stones. The number of stones removed each turn must be one less than a prime number. The winner is the player who takes the last stone. Prove that there are infinitely many values of n for which Bob has the winning strategy.

8. (Putnam 2008, A2). Alan and Barbara play a game in which they take turns filling an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when the entire array is filled. Alan wins if the determinant of the resulting matrix is non-zero; Barbara wins if it is zero. Which player has a winning strategy?

9. (Putnam 1995, B5). The game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking either

(a) One bean from a heap, provided at least two beans are left behind in that heap; or

(b) A complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.