Putnam Club Week Two – 25 September 2012

# Polynomials and Algebra

**1.** Let $k$ be the smallest positive integer for which there exist distinct integers $m_1, m_2, m_3, m_4, m_5$ such that the polynomial

$$P(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly $k$ nonzero coefficients. Find, with proof, a set of such integers $m_1, m_2, m_3, m_4, m_5$ for which this minimum is achieved. (Putnam 1985)

**2.** Find all polynomials $f(x)$ such that $xf(x-1) = (x+1)f(x)$.

**3.** Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

**4.** Prove that if $x^3 + px^2 + qx + r = 0$ has three real solutions, then $p^2 \ge 3q$.

**5.** For which real numbers $c$ is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points? (Putnam 1994)

**6.** Let $f(x)$ be a monic polynomial with integer coefficients, and let $k \in \mathbb{Z}$, $p \in \mathbb{N}$. Prove that if none of the numbers $f(k), f(k+1), \ldots, f(k+p)$ is divisible by $p$, then $f(x) = 0$ has no rational solution.

**7.** Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$\begin{aligned} A &= \left\{ (x,y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\}, \\ B &= \left\{ (x,y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\}, \\ C &= \left\{ (x,y) : x^3 - 3xy^2 + 3y = 1 \right\}, \\ D &= \left\{ (x,y) : 3x^2 y - 3x - y^3 = 0 \right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$. (Putnam 1987)

**8.** Let $p(x)$ be a polynomial with integer coefficients. Suppose that there are three different integers $a, b, c$ such that $p(a) = p(b) = p(c) = -1$. Show that $p(x)$ has no integer roots.

**9.** Let $P(x)$ ba a monic polynomial with integer coefficients. Suppose that there are four different integers $a, b, c, d$ such that $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer $k$ such that $P(k) = 8$. (Canada MO 1970)

**10.** Let $P(x)$ be a polynomial of degree $n$, not necessarily with integer coefficients. For how many consecutive integers must $P(x)$ be an integer in order to guarantee that $P(x)$ is an integer for each integer $x$?