1. Let $k$ be the smallest positive integer for which there exist distinct integers $m_1, m_2, m_3, m_4, m_5$ such that the polynomial

$$P(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

has exactly $k$ nonzero coefficients. Find, with proof, a set of such integers $m_1, m_2, m_3, m_4, m_5$ for which this minimum is achieved. (Putnam 1985)

2. Find all polynomials $f(x)$ such that $xf(x - 1) = (x + 1)f(x)$.

3. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

4. Prove that if $x^3 + px^2 + qx + r = 0$ has three real solutions, then $p^2 \geq 3q$.

5. For which real numbers $c$ is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points? (Putnam 1994)

6. Let $f(x)$ be a monic polynomial with integer coefficients, and let $k \in \mathbb{Z}$, $p \in \mathbb{N}$. Prove that if none of the numbers $f(k), f(k + 1), \ldots, f(k + p)$ is divisible by $p$, then $f(x) = 0$ has no rational solution.

7. Curves $A, B, C$ and $D$ are defined in the plane as follows:

$$A = \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\},$$

$$B = \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\},$$

$$C = \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\},$$

$$D = \left\{(x, y) : 3x^2y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$. (Putnam 1987)
8. Let \( p(x) \) be a polynomial with integer coefficients. Suppose that there are three different integers \( a, b, c \) such that \( p(a) = p(b) = p(c) = -1 \). Show that \( p(x) \) has no integer roots.

9. Let \( P(x) \) be a monic polynomial with integer coefficients. Suppose that there are four different integers \( a, b, c, d \) such that \( P(a) = P(b) = P(c) = P(d) = 5 \). Prove that there is no integer \( k \) such that \( P(k) = 8 \). (Canada MO 1970)

10. Let \( P(x) \) be a polynomial of degree \( n \), not necessarily with integer coefficients. For how many consecutive integers must \( P(x) \) be an integer in order to guarantee that \( P(x) \) is an integer for each integer \( x \)?