Putnam Club Week Four – 9 October 2012

# Calculus

**1.** Let $f$ and $g$ be (real-valued) functions defined on an open interval containing 0, with $g$ nonzero and continuous at 0. If $fg$ and $f/g$ are differentiable at 0, must $f$ be differentiable at 0? (Putnam 2011)

**2.** Evaluate $\displaystyle\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where $a$ and $b$ are positive. (Putnam 1989)

**3.** Let $s$ be any arc of the unit circle lying entirely in the first quadrant. Let $A$ be the area of the region lying below $s$ and above the $x$-axis and let $B$ be the area of the region lying to the right of the $y$-axis and to the left of $s$. Prove that $A + B$ depends only on the arc length, and not on the position, of $s$. (Putnam 1998)

**4.** Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has exactly $n$ distinct real roots? (Putnam 1990)

**5.** Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\, dt$. Evaluate

$$\lim_{n\to\infty} \frac{n!F_n(1)}{\ln n}.$$

(Putnam 2008)

**6.** Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x)\, dx \right| \leq \frac{1}{8} \max_{0\leq x\leq 1} |f'(x)|.$$

(Putnam 2007)

**7.** Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number $q$, the number $f(q)$ is rational and has the same denominator as $q$. (The denominator of a rational number $q$ is the unique positive integer $b$ such that $q = a/b$ for some integer $a$ with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.) (Putnam 2008)