Putnam Club Week Five – 16 October 2012

Games and Algorithms

1. Does the first player in tic-tac-toe have a strategy that guarantees that he does *not* win?

2. Two-move chess is played exactly like ordinary chess, except that during each turn, the current player gets to make two moves (according to the rules of chess) instead of one. (All rules about checkmate, stalemate, draw from repetition of position three times, etc. are adapted accordingly.) Show that in two-move chess, White can, with perfect play, guarantee that he does not lose (i.e., either White wins or the game is a draw).

3. Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.) (Putnam 2008)

4. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (Putnam 2006)

5. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero, Barbara wins if it is zero. Which player has a winning strategy? (Putnam 2008)

6. There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls? (Putnam 2010)

7. A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking **either**

a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy. (Putnam 1995)

8. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$ and replace them with $2x$ and $y - x$. (Putnam 1993)