Putnam Club Week Eight – 6 November 2012

Linear Algebra

1. Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \dots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$. (Putnam 2009)

2. For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, while every dot product of two different rows is odd? (Putnam 2011)

3. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how? (Putnam 2002)

4. The n students at a university form k clubs. A student can be a member of any number of clubs, subject to the two rules that every club must have odd size, and that any two different clubs must have an even number of members in common. Show that $k \leq n$.

5. Let A be an $n \times n$ matrix, and suppose that for every $n \times n$ matrix B, $I + AB$ is invertible. Show that $A = \mathbf{0}$.

6. A *Hadamard Matrix* is a square matrix, all of whose entries are ± 1, with the property that any two different rows disagree in exactly half of their entries (in particular this implies that the dimension of the matrix must be even). If H is an $n \times n$ Hadamard matrix, what can you say about $\det(H)$?

7. Define $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$. Prove that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for all $n \geq 1$.