

## Putnam team training 11/11

- (1) In a  $120 \times 150$  rectangle (made out of unit squares joined along their sides), how many unit squares does its diagonal pass through?
- (2) Let  $\alpha \in \mathbb{R}$ . Find all polynomials  $f$  of degree at most 3 that commute with  $p(x) = x^2 - \alpha$ , i.e. such that  $f(p(x)) = p(f(x))$  for all  $x$ .
- (3) Find all integer solutions to  $xy + 3x - 5y = -3$ .
- (4) Find all positive integers  $x, y$  such that  $4^x + 5 = 9^y$ .
- (5) Let  $a$  and  $b$  be positive integers. If  $\sqrt{a}$  is irrational, prove that  $\sqrt{a} + \sqrt{b}$  is irrational.
- (6) On a table there are 100 tokens. Taking turns two players remove 5, 6, 7, 8, 9 or 10 tokens, at their choice. The player that removes the last token wins. Find a winning strategy and determine which player will be the winner.
- (7) If  $x \neq 0$  prove that

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \cos(x/2^n).$$

- (8) Let  $P_1(x) = x^2 - 2$  and  $P_j(x) = P_1(P_{j-1}(x))$  for  $j \geq 2$ . Prove that for any  $n \in \mathbb{N}$  all roots of the equation  $P_n(x) = x$  are real and distinct.