1. Let $d_n$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example, 

$$
\begin{vmatrix}
\cos 1 & \cos 2 & \cos 3 \\
\cos 4 & \cos 5 & \cos 6 \\
\cos 7 & \cos 8 & \cos 9 
\end{vmatrix}
$$

The argument of $\cos$ is always in radians, not degrees.) Evaluate $\lim_{n \to \infty} d_n$. (Putnam 2009)

2. Alan and Barbara play a game in which they take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy? (Putnam 2008)

3. Let $H$ be an $n \times n$ matrix all of whose entries are $\pm 1$ and whose rows are mutually orthogonal. Suppose $H$ has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$. (Putnam 2005)

4. Let $A$ and $B$ be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible? (Putnam 1991)

5. Let $S$ be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every $a$ in $S$ the set $\{a^n : n = 1, 2, 3, \ldots \}$ is finite. Must $S$ be a group? (Putnam 1989)

6. Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any three (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication. (Putnam 1995)

7. Let $G$ be a group with identity $e$ and $\phi : G \to G$ a function such that

$$
\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)
$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$). (Putnam 1997)