10/12/11 – Polynomials

1. If a and b are the roots of $x^2 + 4x + 7 = 0$, find $a^3 + b^3$.

2. Assume that $x_1, x_2, \ldots, x_7$ are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

(AIME 1989)

3. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν.) (Putnam 2005)

4. Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$. (Putnam 1998)

5. Find all polynomials $p(x)$ such that $p(nm + 1) = p(n)p(m) + 1$ for all integers n, m.

6. Let $p(x)$ be a polynomial with integer coefficients such that $p(0)$ and $p(1)$ are both odd. Show that $p(x)$ has no integer roots.