10/26/11 – Calculus (Week 1)

1. A not uncommon calculus mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \(f(x) = e^{x^2}\), determine, with proof, whether there exists an open interval \((a, b)\) and a nonzero function \(g\) defined on \((a, b)\) such that this wrong product rule is true for \(x\) in \((a, b)\). (Putnam 1988)

2. Is there an infinite sequence of real numbers \(a_1, a_2, a_3, \ldots\) such that
\[
a_1^n + a_2^n + a_3^n + \cdots = m
\]
for every positive integer \(m\)? (Putnam 2010)

3. Suppose that a sequence \(a_1, a_2, a_3, \ldots\) satisfies \(0 < a_n \leq a_{2n} + a_{2n+1}\) for all \(n \geq 1\). Prove that the series \(\sum_{n=1}^{\infty} a_n\) diverges. (Putnam 1994)

4. Functions \(f, g, h\) are differentiable on some open interval around 0 and satisfy the equations and initial conditions
\[
\begin{align*}
    f' &= 2f^2gh + \frac{1}{gh}, \quad f(0) = 1, \\
    g' &= fg^2h + \frac{4}{fh}, \quad g(0) = 1, \\
    h' &= 3fg^2h + \frac{1}{fg}, \quad h(0) = 1.
\end{align*}
\]
Find an explicit formula for \(f(x)\), valid in some open interval around 0. (Putnam 2009)

5. Find all differentiable functions \(f : \mathbb{R} \to \mathbb{R}\) such that
\[
f'(x) = \frac{f(x + n) - f(x)}{n}
\]
for all real numbers \(x\) and all positive integers \(n\). (Putnam 2010)

6. Let \(f\) be a real function on the real line with continuous third derivative. Prove that there exists a point \(a\) such that
\[
f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.
\]
(Putnam 1998)
7. Find all real-valued continuously differentiable functions $f$ on the real line such that for all $x$,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2] \, dt + 1990.$$ 

(Putnam 1990)

8. Let $f$ be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -x g(x) f'(x),$$

where $g(x) \geq 0$ for all real $x$. Prove that $|f(x)|$ is bounded. (Putnam 1997)