10/5/11 – Elementary Number Theory

1. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer for $n > 1$.

2. What is the last digit of

$$7^{(7^{(7^{(7^{(7^{(7^{7})})})})})}?$$

3. Let the Fibonacci numbers f_n be given by $f_1 = 1$, $f_2 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Show that if $n \mid m$, then $f_n \mid f_m$.

4. Define $T_1 = 2$ and $T_{n+1} = T_n^2 - T_n + 1$ for $n \geq 1$. Prove that T_n and T_m are relatively prime for $n \neq m$.

5. Find all non-negative integral solutions to

$$x_1^4 + x_2^4 + \cdots + x_{14}^4 = 1599.$$

(USAMO 1979)

6. (a) Thirteen integers have the property that no matter which one of them you remove, the rest can be divided into two sets of six with the same sum. Prove that all the numbers are the same.

Generalize to (b) rational numbers, and (c) real numbers (harder).