1. Let $f$ be a nonconstant polynomial with positive integer coefficients. Prove that if $n$ is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$. (Putnam 2007)

2. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$  
(Putnam 2009)

3. Prove that the expression

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. (Putnam 2000)

4. Let $N_n$ denote the number of ordered $n$-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether $N_{10}$ is even or odd. (Putnam 1997)

5. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006. (Putnam 2006)

6. Prove that there are unique positive integers $a, n$ such that $a^{n+1} - (a+1)^n = 2001$. (Putnam 2001)

7. Show that for each positive integer $n$,

$$n! = \prod_{i=1}^{n} \text{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$  
(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.) (Putnam 2003)

8. Prove that for every positive integer $n$, there exists an $n$-digit number divisible by $5^n$ all of whose digits are odd. (USAMO 2003)