9/21/11 – The Pigeonhole Principle

1. Seven points are chosen in a regular hexagon of side length 1 unit. Show that some two of them are at most 1 unit apart.

2. Show that of any $n+1$ distinct integers between 1 and $2n$ (inclusive), some pair of them are relatively prime.

3. Show that of any $n+1$ integers between 1 and $2n$, there are two such that one divides the other.

4. Let α be an irrational number. Show that for each n, there is some positive integer multiple of α which is within $\frac{1}{n}$ of an integer.

5. Given any five points on a sphere, show that some four of them must lie on a closed hemisphere. (Putnam 2002)

6. Fifteen positive integers are chosen less than 100. Show that there must be two distinct pairs of these integers whose differences are the same. (That is, there are $a < b \leq c < d$ in the set such that $b - a = d - c$.)

7. 10 guests sit down at a round table at a dinner party. They notice that none of them is sitting at his or her assigned seat. Show that there is some rotation of their seating arrangement such that at least two people are in their assigned seats.

8. Let $x_1, x_2, \ldots, x_6$ be any integers. Show that $\prod_{1 \leq m < n \leq 6} (x_n - x_m)$ is divisible by 5!.

9. Some children work on some math problems, and each child solves at least three problems. Show that either some pair of children solved the same number of problems, or some pair of problems was solved by the same number of children.

10*. Let R be a region in the plane of area greater than 1. Show that there is a translate of R that contains at least two lattice points.