Set theory

1. Given $2^{n-1}$ subsets of a set with $n$ elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.

2. Let $X$ be a subset of \{1, 2, 3, \ldots, 2n\} with $n+1$ elements. Show that we can find $a, b \in X$ with $a$ dividing $b$.

3. Let $S$ be a finite set, and suppose that a collection $F$ of subsets of $S$ has the property that any two members of $F$ have at least one element in common, but $F$ cannot be extended (while keeping this property). Prove that $F$ contains just half of the subsets of $S$.

4. Let $S$ be a set of ordered triples $(a, b, c)$ of distinct elements of a finite set $A$. Suppose that
   (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
   (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ (for $a, b, c$ distinct);
   (c) $(a, b, c)$ and $(c, d, a)$ are both in $S$ if and only if $(b, c, d)$ and $(d, a, b)$ are both in $S$.
   Prove that there exists a one-to-one function $g$ from $A$ to $R$ such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.

5. Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any three (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T$, $U$ is closed under multiplication.

Geometric combinatorics

1. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller $k$?

2. Show that if the points of the plane are colored black or white, then there exists an equilateral triangle whose vertices are colored by the same color.

3. Given a set $M$ of $n \geq 3$ points in the plane such that any three points in $M$ can be covered by a disk of radius 1, prove that the entire set $M$ can be covered by a disk of radius 1.

4. Given that $A, B, \text{ and } C$ are noncollinear points in the plane with integer coordinates such that the distances $AB, AC, \text{ and } BC$ are integers, what is the smallest possible value of $AB$?

5. Is it possible to place infinitely many points in the plane in such a way that all pairwise distances have integer values and points are noncollinear?