Fall 2020

Problem set 1

Friday, September 18th, 2020

Mihaela Ifrim

The following are problems that do not require a lot of known things. However, it will be nice if you would read chapters 1 and 2 of the book *Putnam and Beyond*! Enjoy!!!

1. On a chessboard of infinite dimensions, a rock moves, alternately horizontally and vertically. The rock moves one square on the first move, two squares on the second move, and generally, n squares on the n-th move, for any $n \in \mathbb{N}/\{0\}$. We denote by T the set of all natural numbers n for which there exists a sequence of n moves after which the rock returns to the initial position.

   a) Show that $2013 \notin T$.

   b) What is the cardinality of the set $T \cap \{1, 2, \cdots, 2012\}$.

2. Find $x > 0$, x a real number, and find $n \in \mathbb{N}/\{0\}$ for which

$$[x] + \left\{\frac{1}{x}\right\} = 1.005 \cdot n.$$

   [Hint: here $[x]$ is the integer part of x, and $\left\{\frac{1}{x}\right\}$= the decimals of $\frac{1}{x}$.]

3. We define M to be a *special set* if M is a set consisting of real numbers that satisfy the following properties:

   a) For any $x, y \in M$, $x \neq y$, the numbers $x + y$ and xy are nonzero and exactly one of them is a rational number,

   b) For any $x \in M$, x^2 is irrational.

   What is the maximum number of elements of M?

4. Show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^m} < m$$

   for any $m \in \mathbb{N}/\{0\}$.

5. This problem relates to the one above. Let $p_1, p_2, \cdot p_n$ be prime numbers smaller than 2^{100}. Prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} < 10.$$