Factors of a polynomial

**Theorem.** Let $P(x_1, \ldots, x_n), Q(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ be two polynomials in $n$ variables. Here $\mathbb{R}$ can be replaced by any other field. Suppose $Q(x_1, \ldots, x_n)$ is irreducible, and suppose $P(x_1, \ldots, x_n) = 0$ whenever $Q(x_1, \ldots, x_n) = 0$. Then $P(x_1, \ldots, x_n)$ is divisible by $Q(x_1, \ldots, x_n)$. In other words, $P(x_1, \ldots, x_n)Q(x_1, \ldots, x_n)$ is a polynomial. Moreover, if both of $P, Q$ are of integer coefficients, and if the gcd of the coefficients of $Q$ is 1, then $\frac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)}$ has integer coefficients.

**Example.** Given a polynomial $P(x, y, z)$, prove that the polynomial
\[ Q(x, y, z) = P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x) \]
is divisible by $(x - y)(y - z)(z - x)$.

Viète’s relations

From the fundamental theorem of algebra, it follows that a polynomial with complex number coefficients
\[ P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad (a_n \neq 0) \]
can be factored as
\[ P(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_n). \]
Equating the coefficients of $x$ in the two expressions, we obtain
\[ x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n} \]
\[ x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n = \frac{a_{n-2}}{a_n} \]
\[ \cdots \]
\[ x_1 x_2 \cdots x_n = (-1)^n \frac{a_0}{a_n}. \]

**Example.** If the quartic $x^4 + 3x^3 + 11x^2 + 9x + A$ has roots $k, l, m, n$ such that $kl = mn$, find $A$.

More exercises about polynomials

The problems are not necessarily related to the above two methods.

1. Find all polynomials satisfying the functional equation
\[ (x + 1)P(x) = (x - 6)P(x + 1). \]

2. Let $a, b, c$ be real numbers. Show that $a \geq 0, b \geq 0$ and $c \geq 0$ if and only if $a + b + c \geq 0, ab + bc + ca \geq 0$ and $abc \geq 0$. 


3. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros

$$x_1 < x_2 < \cdots < x_{n-1} < x_n$$

are real. Prove that

$$P'(\frac{x_1 + x_2}{2}) \cdot P'(\frac{x_{n-1} + x_n}{2}) \neq 0.$$ 

4. Let $x_1$ and $x_2$ be the roots of the equation $x^2 + 3x + 1 = 0$. Compute

$$\left(\frac{x_1}{x_2 + 1}\right)^2 + \left(\frac{x_2}{x_1 + 1}\right)^2.$$ 

5. If $x^2 + \frac{1}{x^2} = 14$ and $x > 0$, what is the value of $x^5 + \frac{1}{x^5}$?

6. In $x^3 + px^2 + qx + r = 0$, one zero is the sum of the other two. What is the relation between $p, q$ and $r$?

7. Prove that if $P(x)$ is a polynomial with integer coefficients, and there exists a positive integer $k$ such that none of the integers $P(1), P(2), \ldots, P(k)$ is divisible by $k$, then $P(x)$ has no integral root.

8. Suppose that the function $f(x) = ax^2 + bx + c$, where $a, b, c$ are real constants, satisfies the condition $|f(x)| \leq 1$ for $|x| \leq 1$. Prove that $|f'(x)| \leq 4$ for $|x| \leq 1$.

9. Let $P(x)$ be a cubic polynomial with integer coefficients with leading coefficient 1. Suppose one of its roots is equal to the product of the other two. Show that $2P(-1)$ is a multiple of $P(1) + P(-1) - 2(1 + P(0))$.

10. If $x + y + z = 0$, prove that

$$\frac{x^2 + y^2 + z^2}{2} \cdot \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}.$$ 

11. Let $P(x)$ be a polynomial of degree $n$. Knowing that

$$P(k) = \frac{k}{k+1}, \quad k = 0, 1, \ldots, n,$$

find $P(M)$ for $m > n$.

12. Prove that there are unique positive integers $a, n$ such that

$$a^{n+1} - (a + 1)^n = 2001.$$