Determining closed form expressions for sequences

Useful concepts:

- Pattern recognition and induction.
- Greatest integer function:

\[ \lfloor x \rfloor := \max \{ z \in \mathbb{Z} : z \leq x \} \]

1. Consider the sequence \((a_i)\) given by

\[ a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n}) \]

where \(m \geq n \geq 0\). Find a formula for \(a_n\) if \(a_1 = 1\).

2. Find a formula for the general term of the sequence

\[ 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, \ldots \]

Recursive Sequences

Useful Concept: Characteristic Equation. Let \(x_n = \sum_{i=1}^{k} a_i x_{n-i}\) for some \(k \leq n\). Interpret the equation as the component description of a matrix vector product to find a matrix \(A\) so that \(v_n = A^n v_0\) for some vector \(v_n\) which describes the sequence \(x_n\). The characteristic polynomial of \(A\) is:

\[ P(\lambda) = \sum_{i=0}^{k} -a_i \lambda^{k-i} \]

where \(a_0 = -1\).

Let \(\{\lambda_i\}_{i=1}^{t}\) be the roots of \(P\) (the eigenvalues of \(A\)) with multiplicity \(m_i\). Then

\[ x_n = \sum_{i=1}^{t} \sum_{j=0}^{m_i-1} c_{ij} \binom{n}{j} \lambda_i^{n-j} \]

for some constants \(c_{ij}\). In the case that \(m_i = 1\) this becomes:

\[ x_n = \sum_{i=1}^{k} c_i \lambda_1^n. \]

1. Find the general term of the sequence given by \(x_0 = 3, x_1 = 4\), and

\[ (n + 1)(n + 2)x_n = 4(n + 1)(n + 3)x_{n-1} - 4(n + 2)(n + 3)x_{n-2} \]

for \(n \geq 2\).

2. Consider the sequences

\[ a_0 = 1, \quad a_{n+1} = \frac{3a_n + \sqrt{5a_n^2 - 4}}{2} \]

\[ b_0 = 0, \quad b_{n+1} = a_n - b_n. \]

Prove that \((a_n)^2 = b_{2n+1}\) for all \(n\).
Limits of sequences

Useful concepts:

- Classic analytic definition of limit: For all \( \varepsilon > 0 \) there is an \( N \) so that \( n \geq N \Rightarrow |x_n - L| < \varepsilon \). Consider how this changes for a limit equal to infinity.

- Squeeze theorem: \( a_n \leq b_n \leq c_n, \ a_n \to L, \ c_n \to L \) then \( b_n \to L \). Consider also a version which shows that a limit is infinite.

- Bounded and monotone means convergent.

- Cauchy criterion: Let \( x_n \) be a sequence in a complete metric space (e.g., \( \mathbb{R}^n \)) then \( x_n \) is convergent if and only if for all \( \varepsilon > 0 \) there is \( N \) so that \( n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon \).

- Cesàro-Stolz theorem (discrete analog to L'Hôpital): Let \( x_n \) and \( y_n \) be two real sequences with \( y_n \) positive, increasing, and unbounded. Then
  \[
  \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \to L \Rightarrow \frac{x_n}{y_n} \to L.
  \]

- Nested intervals: Let \( I_k \) be a sequence of closed intervals with \( I_k \supset I_{k+1} \) and diameter of \( I_k \) converging to zero. Then \( \bigcap I_k \) is one point.

1. Let \( x_n \) be a sequence with the property that \( x_n = n^4 \) for all \( n \geq 1 \). Prove that \( x_n \to \infty \).

2. Prove that
  \[ n^2 \int_0^{\frac{1}{2}} x^{n+1} \, dx \to \frac{1}{2} \]

3. Let \( a \) be a positive real number and \( x_n \) a sequence with \( x_1 = a \) and
  \[ x_{n+1} \geq (n + 2)x_n - \sum_{k=1}^{n-1} kx_k. \]

  Find the limit of \( x_n \).

4. Show that
  \[ a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n + 1) \]

  is convergent.

5. Let
  \[ a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2}, \quad c_{n+1} = \frac{c_n + a_n}{2} \]

  Assuming given values for \( a_0, b_0, c_0 \) show that all three sequences converge and find their limits.

6. Let \( t \) and \( \varepsilon \) be real numbers with \( |\varepsilon| < 1 \). Then \( x - \varepsilon \sin x = t \) has a unique real solution.

7. Let \( c, x_0 \) be fixed positive real numbers. Then
  \[ x_n = \frac{1}{2} \left( x_{n-1} + \frac{c}{x_{n-1}} \right) \to \sqrt{c}. \]

8. Let \( k \) be an integer larger than one. Suppose \( a_0 > 0 \) and define:
  \[ a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}. \]

  Evaluate
  \[ \lim_{n \to \infty} \frac{a_{n+1}^k}{n}. \]

9. Let \( f : [a, b] \to [a, b] \) be an increasing function. Show that there is \( c \in [a, b] \) so that \( f(c) = c \).