SECOND ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Find all pairs of positive integers (a, b) such that

$$a^b = b^a.$$

2. Prove that the integral

$$\int_0^\infty \cos(t^3)\, dt$$

converges. (In other words, that the limit

$$\lim_{x \to +\infty} \int_0^x \cos(t^3)\, dt$$

exists.)

3. Find a simple general formula for the expression

$$a_n = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$$

4. Let A be an $n \times n$ matrix with real entries. Suppose that $A^k = A^m$ for some positive integers $k \neq m$. Show that

$$-n \leq \operatorname{tr}(A) \leq n.$$

(Recall that $\operatorname{tr}(A)$ is the sum of A's diagonal entries.)

5. $f(x)$ is a differentiable function satisfying the following conditions:

$$0 < f(x) < 1 \qquad \text{for all } x \text{ on the interval } 0 \leq x \leq 1$$

$$0 < f'(x) < 1 \qquad \text{for all } x \text{ on the interval } 0 \leq x \leq 1.$$

How many solutions does the equation

$$\underbrace{f(f(f \ldots f}_{2016 \text{ times}}(x) \ldots) = x$$

have on the interval $0 \leq x \leq 1$?

(Two more problems on the next page.)

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6. Let p be a prime number. Suppose $\vec{v}_1, \dots \vec{v}_{p+2}$ are $p+2$ planar vectors with integer components. Given two of the vectors $\vec{v}_i$, $\vec{v}_j$, we let $A(\vec{v}_i, \vec{v}_j)$ be the area of the parallelogram spanned by them; recall that it is given by the formula

$$A(\vec{v}_i, \vec{v}_j) = |x_i y_j - y_i x_j| \qquad \text{for} \quad \vec{v_i} = \langle x_i, y_i \rangle, \quad \vec{v_j} = \langle x_j, y_j \rangle.$$

Prove that there are i and j such that $1 \le i \le p+2$, $1 \le j \le p+2$, $i \ne j$, and p divides $A(\vec{v}_i, \vec{v}_j)$.

7. N points are chosen randomly on the unit circle $x^2 + y^2 = 1$. What is the probability that the convex polygon with vertices at these N points (their *convex hull*) contains the origin?