FOURTH ANNUAL UW MADISON UNDERGRADUATE MATH COMPETITION

1. Angela flips a fair coin 2018 times and Bert flips a fair coin 2017 times. What is the probability that Angela had more tails than Bert?

2. Consider a set \( S \) and a binary operation \( \# \), i.e., for each \( a, b \in S \), \( a \# b \in S \). Assume \( (a \# b) \# a = b \) for all \( a, b \in S \). Prove that \( a \# (b \# a) = b \) for all \( a, b \in S \).

3. Let \( n \) be an odd positive integer. Show that the sum

\[
1^n + 2^n + \cdots + n^n
\]

is divisible by \( n^2 \).

4. Each of the six faces of a die is marked with an integer, not necessarily positive. The die is rolled 1000 times. Show that there is a time interval such that the product of all rolls in this interval is a cube of an integer. (For example, it could happen that the product of all outcomes between 5th and 20th throws is a cube; obviously, the interval has to include at least one throw!)

5. For which real numbers \( c \) does the inequality

\[
e^{cx^2} \geq \frac{e^x + e^{-x}}{2}
\]

hold for all real \( x \)?

6. Let \( b_n \) be the sequence of all positive integers such that the decimal expression for \( \frac{1}{b_n} \) terminates in an odd digit:

\[
1, 2, 4, 8, 10, \ldots
\]

(For instance, 3 is not included because \( \frac{1}{3} = 0.33\ldots \) does not terminate, 4 is included because \( \frac{1}{4} = 0.25 \) terminates in 5, which is odd; 5 is not included because \( \frac{1}{5} = 0.2 \) terminates in 2, which is even.)

Find

\[
\sum \frac{1}{b_n}
\]

7. Compute

\[
\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \arctan \left( \frac{1+x^2}{1+y^2} \right) \, dx \, dy
\]

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