38th Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 22, 2016

Fill out the individual registration form

1. Evaluate \( \int_{1}^{2} \frac{\ln x}{2 - 2x + x^2} \, dx \).

2. Determine the real numbers \( k \) such that \( \sum_{n=1}^{\infty} \left( \frac{(2n)!}{4^n n!} \right)^k \) is convergent.

3. Let \( n \) be a positive integer and let \( M_n(\mathbb{Z}_2) \) denote the \( n \) by \( n \) matrices with entries from the integers mod 2. If \( n \geq 2 \), prove that the number of matrices \( A \) in \( M_n(\mathbb{Z}_2) \) satisfying \( A^2 = 0 \) (the matrix with all entries zero) is an even positive integer.

4. For a positive integer \( a \), let \( P(a) \) denote the largest prime divisor of \( a^2 + 1 \). Prove that there exist infinitely many triples \( (a, b, c) \) of distinct positive integers such that \( P(a) = P(b) = P(c) \).

5. Suppose that \( m, n, r \) are positive integers such that 
\[
1 + m + n\sqrt{3} = (2 + \sqrt{3})^{2r-1}.
\]
Prove that \( m \) is a perfect square.

6. Let \( A, B, P, Q, X, Y \) be square matrices of the same size. Suppose that
\[
A + B + AB = XY \quad \quad \quad AX = XQ \\
P + Q + PQ = YX \quad \quad \quad PY = YB.
\]
Prove that \( AB = BA \).

7. Let \( q \) be a real number with \( |q| \neq 1 \) and let \( k \) be a positive integer. Define a Laurent polynomial \( f_k(X) \) in the variable \( X \), depending on \( q \) and \( k \), by
\[
f_k(X) = \prod_{i=0}^{k-1} (1 - q^i X)(1 - q^{i+1} X^{-1}). \]
(Here \( \prod \) denotes product.) Show that the constant term of \( f_k(X) \), i.e. the coefficient of \( X^0 \) in \( f_k(X) \), is equal to
\[
\frac{(1 - q^{k+1})(1 - q^{k+2}) \cdots (1 - q^{2k})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}.
\]