38th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 22, 2016

Fill out the individual registration form

1. Evaluate $\int_1^2 \frac{\ln x}{2-2x+x^2}\,dx$.

2. Determine the real numbers k such that $\sum_{n=1}^{\infty}\left(\frac{(2n)!}{4^n n! n!}\right)^k$ is convergent.

3. Let n be a positive integer and let $\mathrm{M}_n(\mathbb{Z}_2)$ denote the n by n matrices with entries from the integers mod 2. If $n \geq 2$, prove that the number of matrices A in $\mathrm{M}_n(\mathbb{Z}_2)$ satisfying $A^2 = 0$ (the matrix with all entries zero) is an even positive integer.

4. For a positive integer a, let $P(a)$ denote the largest prime divisor of $a^2 + 1$. Prove that there exist infinitely many triples (a,b,c) of distinct positive integers such that $P(a) = P(b) = P(c)$.

5. Suppose that m,n,r are positive integers such that
$$1+m+n\sqrt{3} = (2+\sqrt{3})^{2r-1}.$$
Prove that m is a perfect square.

6. Let A,B,P,Q,X,Y be square matrices of the same size. Suppose that
$$A+B+AB = XY \qquad AX = XQ$$
$$P+Q+PQ = YX \qquad PY = YB.$$
Prove that $AB = BA$.

7. Let q be a real number with $|q| \neq 1$ and let k be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable X, depending on q and k, by $f_k(X) = \prod_{i=0}^{k-1}(1-q^iX)(1-q^{i+1}X^{-1})$. (Here $\prod$ denotes product.) Show that the constant term of $f_k(X)$, i.e. the coefficient of X^0 in $f_k(X)$, is equal to
$$\frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{2k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$