Math 521 **Midterm 1** **Spring 2013**

NAME:

Grading Table

Question	Possible Points	Points Earned
1	10	
2	15	
3	10	
Total	35	

(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A subset X of $\mathbb{R}$ so that $\mathrm{cl}(X) = \mathbb{R}$ and $\mathrm{int}(X) = \emptyset$.
(b) A non-empty set X in any metric space so that $\partial X = X$.
(c) A subset of $\mathbb{Q}$ which is neither open nor closed.
(d) Two Cauchy sequences in $\mathbb{Q}$: (x_n) and (y_n) so that $\forall k\, x_k < y_k$, but $[(y_n)] \leq [(x_n)]$.
(e) A set $X \subseteq \mathbb{R}^2$ which is open in $\mathbb{R}^2$ and so that $X \cap \mathbb{R}$ is not open in $\mathbb{R}$. (Here I'm using the symbol $\mathbb{R}$ to mean the x-axis of $\mathbb{R}^2$.)

Solutions:

(a) Q
(b) $\{1\}$ in the metric space $\mathbb{R}$. Or the unit circle in $\mathbb{R}^2$.
(c) $[0, 1) \cap \mathbb{Q})$
(d) $x_n = -\frac{1}{n}$, $y_n = \frac{1}{n}$
(e) No example (Proof: the inherritance principle)

(2) (15 points) Recall that a metric space is complete if every Cauchy sequence converges.

(a) Give the definition of a *Cauchy sequence*.

(b) Let $M = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with the distance $d(x,y) = |x-y|$. Let N be $\mathbb{N}$ with the discrete metric. Show that M and N are homeomorphic.

(c) Show that completeness is not a topological property, i.e., show that it is possible for M and N to be homeomorphic, but only one of the two is complete.

Solutions:

(a) (p_n) is Cauchy if $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n, m \in \mathbb{N}(n, m > N \to d(p_n, p_m) < \epsilon)$

(b) Let $f : N \to M$ be defined by $f(x) = \frac{1}{x}$. f is clearly a bijection. We need to show that f and f^{-1} are continuous.

First, let's show that every subset of M is open. Given a point $\frac{1}{n}$, let $r = \frac{1}{n} - \frac{1}{n+1}$. Then $B_r^M(\frac{1}{n}) = \{\frac{1}{n}\}$. Thus each point is open, so every set, being a union of points is open. Now, $f^{pre}(U)$ is open for any U since every subset of N is open. Similarly, $(f^{-1})^{pre}(U)$ is open for any U since every subset of M is open. Thus f and f^{-1} are continuous.

(c) Let's us M and N from the last part. They are homeomorphic. First, M is not complete: (x_n) where $x_n = \frac{1}{n}$ is a Cauchy sequence with no limit in M. Next, let's see that N is complete. Let (x_n) be a Cauchy sequence in N. Then after some N, all terms have distance less than 1. Thus all terms after N are equal. Thus the sequence converges. So every Cauchy seqence in N converges in N.

(3) (10 points) For M a metric space, $p \in M$, and $r \in \mathbb{R}$, Define $C_r(p)$ to be the set of points of distance $\leq r$ to p. That is:

$$C_r(p) = \{q \in M \mid d(p,q) \leq r\}$$

(a) Show that for any $r > 0$ and any $p \in M$, $C_r(p)$ is a closed subset of M.

(b) Is it true that for any metric space M and any $p \in M$, $C_r(p) = \mathrm{cl}(B_r(p))$? Give either a proof or a counter-example.

Solutions:

(a) Let's show that the complement of $C_r(p)$ is open. Fix $q \notin C_r(p)$. Then $d(q,p) > r$. Let $\epsilon = d(q,p) - r$. Let's show that $B_\epsilon(q)$ is entirely contained in the complement of $C_r(p)$. If $x \in B_\epsilon(q)$, then $d(x,q) < \epsilon = d(q,p) - r$. So, $d(q,p) - d(q,x) > r$. Since $d(x,p) + d(q,x) \geq d(q,p)$, we see $d(x,p) \geq d(q,p) - d(q,x) > r$. Thus $x \in B_\epsilon(q)$ implies x in the complement of $C_r(p)$. Thus the complement of $C_r(p)$ is open, and $C_r(p)$ is closed.

(b) Let M be $\mathbb{R}$ with the discrete metric. Let p be any point. Then $\mathrm{cl}(B_r(p)) = \{p\}$, but $C_r(p) = M$. So, they need not be equal.