1. 1.14

For n=1: Let $\delta = \varepsilon$. This works trivially. Induction step: Suppose it's true for n:

$$\forall y \forall \varepsilon > 0 \exists \delta > 0 |u - y| < \delta \to |u^n - y^n| < \varepsilon$$

Let's show it for $n + 1$: Let y and $\varepsilon > 0$ be given. Then

$$|u^{n+1} - y^{n+1}| \le |u - y||u^n + u^{n-1}y + \ldots + y^n|$$
$$\le |u - y|(|u|^n + |u|^{n-1}|y| + \ldots |y|^n)$$
$$\le |u - y|(n \max\{|u|, |y|\}^n)$$

Fix δ so that $\delta < 1$ and $\delta < \frac{\varepsilon}{n(|y|+1)^n}$. If $|u - y| < \delta$, then

$$|u^n - y^n| \le \delta \cdot n \cdot \max\{|u|, |y|\}^n \le \delta \cdot n \cdot (|y| + 1)^n < \frac{\varepsilon}{n(|y| + 1)^n} n \cdot (|y| + 1)^n = \varepsilon$$

2. 1.15

Need to show that for any $x > 0$ there is a unique $y > 0$ with $y^n = x$.

Let $y = \sup(S)$ where $S := \{s \in \mathbb{R} \mid s^n \le x\}$

Claim 2.1. $y^n = x$

Proof. We'll use the ε principle and show $|y^n - x| < \varepsilon$ for every positive ε. Use 14 to get a δ so that $|u - y| < \delta \to |u^n - y^n| < \varepsilon$.

Then y, being $\sup(S)$ is within δ of an element of S and an element of S^C. So, $\exists z \in S$ so $|z - y| < \delta$. Thus $|u^n - y^n| < \varepsilon$, showing that $y^n < \varepsilon + u^n \le \varepsilon + x$ (the last inequality being because $u \in S$.

Similarly, $\exists z \notin S$ so $|z - y| < \delta$. Thus $|u^n - y^n| < \varepsilon$, showing that $y^n > u^n - \varepsilon > x - \varepsilon$. Putting these two inequalities together $|y^n - x| < \varepsilon$. As $\varepsilon > 0$ was arbitrary, $|y^n - x| = 0$. ☐

We now need to show uniqueness. Suppose $0 < a < b$. We will show that $0 < a^k < b^k$ for any k. We show this by induction. The base case, when $k = 1$ is trivial (the hypothesis and conclusion are the same). Suppose it is true for k, that is $0 < a < b$ implies $0 < a^k < b^k$. We now show it for $k + 1$: Using the properties of ordered fields: $0 \cdot a < a^k \cdot a < b^k \cdot a < b^k \cdot b$, so $0 < a^{k+1} < b^{k+1}$, which is what we needed.

3. 1.16

a) I add a bit of notation to make the following more streamlined: $x_0 = N$. By Definition, for $k \ge 0$, x_k is the greatest integer $\le 10^k(x - (N + \frac{x_1}{10} + \ldots \frac{x_{k-1}}{10^{k-1}}))$. So,

$$x_k \le 10^k(x - (N + \frac{x_1}{10} + \ldots \frac{x_{k-1}}{10^{k-1}})) < x_k + 1$$

, so:

$$0 \le 10^k(x - (N + \frac{x_1}{10} + \ldots \frac{x_{k-1}}{10^{k-1}} + \frac{x_k}{10^k})) < 1$$

, so:

$$0 \le 10^{k+1}(x - (N + \frac{x_1}{10} + \ldots \frac{x_{k-1}}{10^{k-1}} + \frac{x_k}{10^k})) < 10$$

As x_{k+1} is the least integer $\le 10^{k+1}(x - (N + \frac{x_1}{10} + \ldots \frac{x_{k-1}}{10^{k-1}} + \frac{x_k}{10^k})) < 10$, we see that x_{k+1} is between 0 and 9. As this works for any $k \ge 0$, we have all x_i are between 0 and 9 for $i \ge 1$.

b) Suppose toward a contradiction that k is the least so that $x_l = 9$ for all $l \geq k$. Then for any $l \geq k$

$$N + \frac{x_1}{10} + \dots \frac{x_{k-1}+1}{10^{k-1}} > x > N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}} + \frac{9}{10^k} + \dots + \frac{9}{10^l}$$

Subtracting from both sides:

$$\frac{1}{10^{k-1}} > x - (N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}}) > \frac{9}{10^k} + \dots + \frac{9}{10^l}$$

, or:

$$0 > x - (N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}} + \frac{1}{10^{k-1}}) > -\frac{1}{10^{l+1}}$$

. Or, considering absolute values,

$$|x - (N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}} + \frac{1}{10^{k-1}})| < \frac{1}{10^{l+1}}$$

for any l. For any positive ε, there is some l so that $\frac{1}{10^{l+1}}$, so $|x - (N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}} + \frac{1}{10^{k-1}})| < \varepsilon$ for any positive ε, so $|x - (N + \frac{x_1}{10} + \dots \frac{x_{k-1}}{10^{k-1}} + \frac{1}{10^{k-1}})| = 0$. But this contradicts the definition of x_{k-1} (it wasn't the greatest integer it could have been). This contradiction shows that there cannot be a final sequence of 9's.

c) Consider the set $S = \{N, N + \frac{x_1}{10}, \dots\}$.

Claim 3.1. $N + 1$ is an upper bound for S.

Proof. Let's consider an element of S:

$$N + \frac{x_1}{10} + \dots + \frac{x_k}{10^k} \leq N + \frac{9}{10} + \dots + \frac{9}{10^k} \leq N + \frac{10^k - 1}{10^k} \leq N + 1$$

□

Let us consider the decimal expansion of x, the least upper bound for S: $M.y_1y_2\dots$.

As $N \leq x < N + 1$ (the second inequality is due to the fact that one of the x_k is < 9): $M = N$.

$x - N$ is the least upper bound for the set $\{0, \frac{x_1}{10}, \frac{x_1}{10} + \frac{x_2}{100}, \dots\}$, so $10(x - N)$ is the least upper bound for the set $\{x_1, x_1 + \frac{x_2}{10}, \dots\}$. So, $x_1 \leq 10(x - N) \leq x_1 + 1$, showing that $x_1 = y_1$. Continuing as such, we see that $x_i = y_i$ for all i.

d) Repeat everything with 10 replaced by a general base.

4. 1.30

a) Define $f(x+) = \inf\{f(y) \mid y > x\}$ and $f(x-) = \sup\{f(y) \mid y < x\}$, and the jump of f at x to be $f(x+) - f(x-)$.

Claim 4.1. f is continuous at x if and only if the jump of f at x is 0.

Proof. $\to$: We'll show that the jump is $< \varepsilon$ for each positive ε. Let $\varepsilon > 0$ be fixed. By continuity, there is a δ so that $|x - y| < \delta \to |f(x) - f(y)| < \frac{\varepsilon}{2}$.

Then $f(x+) \leq f(x + \frac{\delta}{3}) \leq f(x) + \frac{\varepsilon}{2}$, and similarly $f(x-) \geq f(x - \frac{\delta}{3}) \geq f(x) - \frac{\varepsilon}{2}$. So $f(x+) - f(x-) \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, $f(x+) - f(x-) = 0$.

$\leftarrow$: Suppose the jump of f at x is 0, and let $\varepsilon > 0$ be given.

As the jump is 0, $f(x+) = f(x-) = c$. Let $d \in \{f(y) \mid y > x\}$ be so $d < c + \varepsilon$, and let $e \in \{f(y) \mid y < x\}$ be so $e > c - \varepsilon$ (these exist by the definition of $f(x+)$ as an inf and $f(x-)$ as a sup). Let $z > x$ be so $f(z) = d$ and $w < x$ be so that $f(w) = e$. Finally, let $\delta = \min\{x - w, z - x\}$. If $|y - x| < \delta$, then either

$w < y < x$, thus $f(w) < f(y) < f(x)$ or $x \leq y < z$, thus $f(x) \leq f(y) < f(z)$. So, $f(w) < f(y) < f(z)$, showing $|f(y) - c| < \varepsilon$. It remains to see that $c = f(x)$.

If it were true that $c < f(x)$, then by the definition of $f(x+)$ as an inf, there would have to be a $z > x$ so that $f(z) < f(x)$ which would contradict f being monotone. Similarly, we can contradict the possibility that $c > f(x)$. Thus $c = f(x)$, and our δ above witnesses continuity of f. □

Note that if $x < y$, then $f(x-) \leq f(x) \leq f(x+) \leq f(y-) \leq f(y) \leq f(y+)$. So, if there were n jumps of size at least q, say at $x_1, \ldots, x_n$, then $f(a) \leq f(x_1-) \leq f(x_1+) - q \leq f(x_2-) - q \leq f(x_2+) - 2q \ldots \leq f(x_n+) - nq \leq f(b)$. It follows that there can be at most $\frac{f(b)-f(a)}{q}$ many jumps of size $\geq q$. Thus the set of points with non-zero jump $\bigcup_{q>0, q\in\mathbb{Q}}\{\text{jumps of size at least } q\}$. This is a countable union of countable sets, so is countable. Thus there are countably many discontinuities of f in the interval $[a, b]$.

b)Yes! Consider f on the intervals $[n, n+1]$ for each $n \in \mathbb{Z}$. On each interval, there are only countably many discontinuities of f by part a. Since there are only countably many such intervals (they are indexed by $\mathbb{Z}$, which is countable), the set of discontinuous points is a countable union of countable sets, so is countable.

5. Multiplication is well-defined

Suppose $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$. We need to show that $(a_n b_n) \sim (a'_n b'_n)$. We use the fact that every Cauchy sequence is bounded. Let $M \in \mathbb{Q}$ be so that $(a_n), (b_n), (a'_n), (b'_n)$ are sequences in $[-M, M]$. Let $\varepsilon > 0$ be given. We need to show that there is some $N \in \mathbb{N}$ so that $n > N$ implies that $|a_n b_n - a'_n b'_n| < \varepsilon$. Let N be large enough that $n > N$ implies $|a_n - a'_n| < \frac{\varepsilon}{4M}$ and $|b_n - b'_n| < \frac{\varepsilon}{4M}$. Then for $n > N$, $|a_n b_n - a'_n b'_n| = |a_n b_n - a_n b'_n + a_n b'_n - a'_n b'_n| \leq |a_n b_n - a_n b'_n| + |a_n b'_n - a'_n b'_n| = |a_n||b_n - b'_n| + |b'_n||a_n - a'_n| < M \cdot \frac{\varepsilon}{4M} + M \cdot \frac{\varepsilon}{4M} < \varepsilon$.

6. First written problem

We need to show that the following two conditions are equivalent:

(1) $\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N}(n > N(\varepsilon) \to a_n > b_n - \varepsilon)$

(2) $(a_n) \sim (b_n)$ OR $\exists N \in \mathbb{N}(n > N \to a_n > b_n)$

Let's first show $1 \to 2$. For each $\varepsilon > 0$, let $K(\varepsilon) > N(\varepsilon)$ be large enough that $n, m > K$ implies $|a_n - a_m| < \frac{\varepsilon}{4}$ and $|b_n - b_m| < \frac{\varepsilon}{4}$. We proceed based on cases:

Case 1: For each $\varepsilon > 0$, There is no $n > K(\varepsilon)$ so that $a_n \geq b_n + \varepsilon$. In this case, $n > K(\varepsilon)$ (thus $n > N(\varepsilon)$) implies that $-\varepsilon < a_n - b_n < \varepsilon$, so $|a_n - b_n| < \varepsilon$. As this works for every $\varepsilon > 0$, we have that $(a_n) \sim (b_n)$, so condition 2 holds.

Case 2: For some $\varepsilon > 0$, there is an $n > K(\varepsilon)$ so that $a_n \geq b_n + \varepsilon$. In this case, we will show that for any $m > K(\varepsilon)$, $a_m > b_m$. For $m > K(\varepsilon)$:

$$a_m - b_m = (a_m - a_n) + (a_n - b_n) + (b_n - b_m) \geq -\frac{\varepsilon}{4} + \varepsilon - \frac{\varepsilon}{4} \geq \frac{\varepsilon}{2} > 0$$

So condition 2 holds.

Now let's show $2 \to 1$. Suppose the first case of condition 2 holds. Then $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ so that $n > N$ implies $|a_n - b_n| < \varepsilon$. So, in particular, $n > N$ implies $a_n > b_n - \varepsilon$.

If the other case holds, for some $N \in \mathbb{N}$, $n > N$ implies that $a_n > b_n$. In particular, it implies $a_n > b_n - \varepsilon$ for any positive ε.

7. Second written problem

We need to show that $[(a_n)] < [(b_n)]$ if and only if $\exists \varepsilon > 0 \exists N \in \mathbb{N}$ so $n > N$ implies $b_n > a_n + \varepsilon$.

Let's first show the rightward implication. Assume $[(a_n)] < [(b_n)]$ and suppose towards a contradiction that the condition on the right fails. That means for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there is some $k > N$ with $a_k \geq b_k - \varepsilon$. By the problem above, after some M, $a_n < b_n$. We now will derive a contradiction by proving $(a_n) \sim (b_n)$. Let $\varepsilon > 0$ be given. Let $N > M$ large enough that $n, m > N$ implies $|a_n - a_m| < \frac{\varepsilon}{4}$ and $|b_n - b_m| < \frac{\varepsilon}{4}$. Let $k > N$ be so that $a_k \geq b_k - \frac{\varepsilon}{2}$. Then for any $m > N$ we have that

$$0 < b_m - a_m = (b_m - b_k) + (b_k - a_k) + (a_k - a_m) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$

Now let's show the leftward implication. By the above problem, it's clear that $[(a_n)] \leq [(b_n)]$. We need only show that $(a_n) \not\sim (b_n)$. Suppose towards a contradiction that $(a_n) \sim (b_n)$. Let ε and N be so that $n > N$ implies $b_n > a_n + \varepsilon$. Let M be so that $n > M$ implies $|a_n - b_n| < \varepsilon$. But then for any $k > N$ and $k > M$, $b_n - a_n > \varepsilon$ and $b_n - a_n < \varepsilon$. This is a contradiction.

8. 2.17

Suppose $f : M \to N$ has the property that it sends convergent sequences to convergent sequences. We need to show that it sends limits to limits.

Claim 8.1. If (p_n) is a sequence in M converging to p, then $(f(p_n))$ is a sequence in N which converges to $f(p)$

Proof. Consider the sequence $(p_1, p, p_2, p, p_3, p, \ldots)$ in M. This is still a convergent sequence with limit p. Thus, by the hypothesis, the sequence $(f(p_1), f(p), f(p_2), f(p) \ldots)$ is a convergent sequence in N. The subsequence $(f(p), f(p), f(p), \ldots)$ must converge to the same limit, thus the limit is $f(p)$. But then the other subsequence $(f(p_1), f(p_2), \ldots)$ must converge to $f(p)$. □

By the sequence convergence condition for continuity, we see that f is continuous.

9. 2.18

a) Let $\delta = \varepsilon$.

b) It is a bijection by definition. Check (the definition) that the inverse of an isometry is an isometry, so by part a, they are both continuous.

c) Suppose $f : [0, 2] \to [0, 1]$ were an isometry. Then $d_{[0,1]}(f(0), f(2)) = 2$. But there are no two points in $[0, 1]$ of distance 2! This is a contradiction, so there can be no isometry between these two spaces.

10. 2.19

Check the definitions: The identity map: $M \to M$ is always an isometry. If $f : M \to N$ is an isometry, so is $f^{-1} : N \to M$. If $f : M \to N$ and $g : N \to K$ are isometries, then $gf : M \to K$ is also an isometry.

11. 2.21

Draw pictures and play

12. 2.22

No, $\mathbb{R}$ and $\mathbb{Q}$ are not even bijective as they have different cardinalities.

13. 2.23

Suppose $f : \mathbb{Q} \to \mathbb{Z}$ were a homeomorphism. Let $f(q) = 0$, and let $\varepsilon = 1$. Which $\delta > 0$ will ensure that $|q - r| < \delta \to |f(q) - f(r) < 1|$? Well, $f(q) - f(r)$ can only be < 1 if it is 0, i.e., if $q = r$. Thus no $\delta > 0$ will suffice. So, f is not continuous at q. This is a contradiction.

Alternatively, use Corollary 11: in $\mathbb{Z}$, every set is open. If there were a homeomorphism with $\mathbb{Q}$, then every subset of $\mathbb{Q}$ would also be open. But this is not true!

14. 2.26

a) Suppose (x_n) converges to the point x in $\mathbb{R}$. We'll see that $(|x_n|)$ converges to the point $|x|$ in $\mathbb{R}$. Let $\varepsilon > 0$ be given. There exists an $N \in \mathbb{N}$ so that $n > N \to |x_n - x| \leq \varepsilon$. But $|x_n| - |x| \leq |x_n - x|$ and $|x| - |x_n| \leq |x_n - x|$ (both by the triangle inequality). So, $||x_n| - |x|| < \varepsilon$, so this same N shows that $(|x_n|)$ converges to $|x|$.

b) Consider the sequence $(-1, 1, -1, 1, \ldots)$. The absolute values converge, but the sequence does not.

15. 2.30

a) Let (p_n) be a sequence which is monotone and bounded. Suppose it increases (decreases is symmetric). Let p be the least upper bound for the set $S = \{p_n \mid n \in \mathbb{N}\}$. We need to show that p_n converges to p. Let $\varepsilon > 0$ be given. Since p is the *least* upper bound for S, there is a p_N in S so that $p - \varepsilon < p_N \leq p$. But then, since the sequence is increasing, $n > N \to p_n > p_N$, so $p - \varepsilon < p_n \leq p$ (the second inequality because $p_n \in S$ and p is an upper bound for S). Thus for any $n > N$, $|p - p_n| < \varepsilon$.

b)We used the least upper bound property to prove the monotone sequence condition (the conclusion of part a). Now let's do the reverse. Suppose S is a bounded set. We'll define a sequence p_n of points in S so that p_n converges to the least upper bound of S.

Let p_1 be any point in S. Having define p_k, we will pick p_{k+1} as follows: Let n be least so that there exists any element of S which is $> p_k + 2^{-n}$. Let p_{k+1} be such an element. If there is no such n, then p_k is in S but there is no greater element of S, so p_k is the least upper bound. So, we're either done, or we have a sequence (p_n) which is increasing. Since it's in S, it's also bounded. So, by the monotone sequence condition, (p_n) converges, say to the point p. We'll show that p is the least upper bound for S.

Suppose p were not an upper bound for S. Then there is a point $a \in S$ which is bigger than p. Let l be so that $2^{-l} < a - p$. As (p_n) converges to p, there is a k so that $p - 2^{-l-1} < p_k < p$. But then there is a point (namely a) of S so that $a > p_k + 2^{-l}$, so we chose p_{k+1} to be such a point. So $p_{k+1} > p_k + 2^{-l} > p - 2^{-l-1} + 2^{-l} > p$. This is a contradiction, since the sequence (p_n) is increasing, so it will never (after $k + 1$) get closer to p than p_{k+1} is.

Suppose p were not the least upper bound for S: Say $y < p$ is an upper bound for S. But since (p_n) converges to p, there is a k so that $|p_k - p| < p - y$. But then $p_k > y$ showing that y isn't an upper bound for S.

16. 2.31

a) Suppose there are infinitely many elements of the sequence which are $\geq x_1$. If not, there are infinitely many elements of the sequence which are $\leq x_1$, and a symmetric approach will work. We'll attempt to define a sequence which is increasing. If we fail, we will build a sequence which is decreasing.

Let $p_1 = x_1$. Pick p_2 to be x_{k_2} where $k_2 > 1$ is the least with the property that $x_{k_2} \geq p_1$ and infinitely many x_l are $\geq x_{k_2}$. Then pick p_3 to be x_{k_3} where $k_3 > k_2$ is the least with the property that $x_{k_3} \geq p_2$ and infinitely many x_l are $\geq x_{k_3}$. If this process continues infinitely, this builds an increasing sequence.

Suppose it fails: At some step, there is no x_{k_n} so that $x_{k_n} \geq x_{k_{n-1}}$ and there are infinitely many $x_l \geq x_{k_n}$. What's this get us? For any $x_{k_n} \geq x_{k_{n-1}}$, there are only finitely many x_l with $x_l > x_{k_n}$. Note that for any x_l for $l > k_{n-1}$ there must be an x_j with $j > l$ so that $x_{k_n} < x_j < x_l$, since only finitely many are larger.

Let's now build a decreasing sequence. Choose q_1 to be any x_l with $l > k_n$ and $x_l > x_{k_n}$. Then there must be an x_j with $j > l$ and $x_{k_n} < x_j < x_l$. Let q_2 be this x_j. Repeating as such, we build a decreasind sequence.

b)Any bounded sequence has a monotone and bounded subsequence (by part a). By 2.30a, this is convergent.

17. 1.39

a) If f is uniformly continuous, that means for every $\varepsilon > 0$ there exists a $\delta > 0$ which simultaneously works for every point x (i.e., δ doesn't even depend on x). This is certainly good enough to see that for every x and every $\varepsilon > 0$ there is a $\delta > 0$ which works. The other way, $f(x) := \sin(\frac{1}{x})$ has the property that in any interval $(0, \delta)$ there is an a and a b so that $f(a) = -1$ and $f(b) = 1$. Thus if any single δ was claimed to work for $\varepsilon = 1$, it would fail for the point $x = \delta$.

b) Yes! For any ε, let $\delta = \frac{\varepsilon}{2}$. (Fill in the details)

c) No! For any δ, you can always find an x so that $(x+\delta)^2 - x^2 > 1$ (say $x = \frac{1}{2\delta}$ will work). This shows that for $\varepsilon = 1$, no δ can work.

18. 2.1

a) $(0, 1)$ is open in $\mathbb{R}$:

Given $p \in (0, 1)$, let $\varepsilon = \min\{p, 1-p\}$. Then $B_{\varepsilon(p)} = (p-\varepsilon, p+\varepsilon) \supseteq (0, 1)$. Since $p \in (0, 1)$, ε is > 0, so this shows that $(0, 1)$ is open.

b) $(0, 1)$ is not open in $\mathbb{R}^2$, where $(0, 1)$ represents the associated interval on the x-axis.

Let $p = \frac{1}{2} \in (0, 1)$. Note that in $\mathbb{R}^2$, p is the point with x-coordinate $\frac{1}{2}$ and y-coordinate 0. Let $\varepsilon > 0$ be given. Then the point with x-coordinate $\frac{1}{2}$ and y-coordinate $\frac{\varepsilon}{2}$ is within the ball $B_\varepsilon(p)$ but is not in the set $(0, 1)$.

19. 2.2

Answer: For the intervals $[a, b]$ where both a and b are irrational numbers.

Claim 19.1. If either a or b is rational, then $[a, b] \cap \mathbb{Q}$ is not open in $\mathbb{Q}$.

Proof. Consider the point p which is either a or b (whichever is rational). Then there is no ε so that $B_\varepsilon(p) \cap \mathbb{Q} \subseteq [a, b]$. $\square$

Claim 19.2. If both a and b are irrational, then $[a, b] \cap \mathbb{Q}$ is clopen in $\mathbb{Q}$.

Proof. Openness: Let p be a point in $[a, b] \cap \mathbb{Q}$. By assumption, p cannot be a or b. Let $\varepsilon = \min\{p - a, b - p\}$. Then $B_\varepsilon(p) \cap \mathbb{Q} \subseteq [a, b] \cap \mathbb{Q}$.

Closedness: We need to show that if $(p_n)_{n\in\mathbb{N}}$ is a sequence in $[a, b] \cap \mathbb{Q}$ which converges to $p \in \mathbb{Q}$, then $p \in [a, b] \cap \mathbb{Q}$. But $[a, b]$ is closed in $\mathbb{R}$, so $p \in [a, b]$ and $p \in \mathbb{Q}$, so $p \in [a, b] \cap \mathbb{Q}$. $\square$

20. 2.3

We need to show that $S = \{p\}$ is closed in M for any $p \in M$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in S. Then since S contains only the point p, the sequence is so that $p_n = p$ for each $n \in \mathbb{N}$. But then the limit of (p_n) is the point p, which is in S. Thus S is closed.

Since the finite union of closed sets is closed, we see that any finite set is closed.

21. 2.5

S is open if and only if S^C is closed. This means all of the limit points of S^C are in S^C. But this just means that all of the limit points of S^C are not in S, or no points of S are limit points of its complement.

22. 2.11

Let f be the function sending an open set U to the closed set U^C. Let g be the function sending a closed set K to the open set K^C. Note that both fg and gf are the identity function, so they are both bijections.

23. 2.12

a) First suppose M is a metric space with the discrete metric. We will show that every set is open, which (by taking complements) shows that every set is closed as well. To see that every set is open, since every set is a union of single points, we need to only see that every set of the form $S = \{p\}$ is open. This is true by taking $\varepsilon = \frac{1}{2}$. $B_\varepsilon(p) = \{p\} \subseteq S$. Thus S is open, and every set is open. As there is a bijection between the topologies of homeomorphic metric spaces, this shows that every subset of any homeomorph of a discrete space is open.

b)Recall that $f : M \to N$ is continuous is $f^{pre}(U)$ is open for every open $U \subseteq N$. But if every subset of M is open, this this condition must hold.

c)Suppose (p_n) converges to the point p. As the set $S = \{p\}$ is open, there is some ε so that $\{p\} \subseteq B_\varepsilon(p) \subseteq \{p\}$, so $B_\varepsilon(p) = \{p\}$. But then there is some $N \in \mathbb{N}$ so that $n > N \to p_n \in B_\varepsilon(p) = \{p\}$, so from some point onwards the sequence is constant. Thus convergent implies constant from some point onwards. Conversely, if a sequence is constant from some point onwards, it is certainly convergent.

24. 2.14

Recall $\text{dist}(p, S) = \inf\{d(p, s) \mid s \in S\}$.

a)

Claim 24.1. For every $p \in M$ and $S \subseteq M$, $p \in lim(S)$ if and only if $\text{dist}(p, S) = 0$

Proof. $\rightarrow$: If p is a limit point of S, then for every $\varepsilon > 0$, there is a point $s \in S$ with $d(p, s) < \varepsilon$. Thus $\text{dist}(p, S) < \varepsilon$. By the ε-principle, $\text{dist}(p, S) \leq 0$. But 0 is a lower bound for $\{d(p, s) \mid s \in S\}$, so $\text{dist}(p, S) \geq 0$, thus $\text{dist}(p, S) = 0$

$\leftarrow$: If $\text{dist}(p, S) = 0$, we'll show that p is a limit point of S. We know it suffices to show that for every $\varepsilon > 0$, $B_\varepsilon(p) \cap S \neq \emptyset$. Let $\varepsilon > 0$ be given. Then $\text{dist}(p, S) = 0 < \varepsilon$ means that ε is not a lower bound for $\{d(p, s) \mid s \in S\}$. Thus there is a point $s \in S$ so that $d(p, s) < \varepsilon$. But then $s \in B_\varepsilon(p) \cap S$, showing that the intersection is non-empty, and thus p is a limit point of S.

□

25. 2.34

$f : M \rightarrow N$ is open if for each open set $U \subseteq M$, the image set $f(U) \subseteq N$ is open in N.

a) Let $f : R \rightarrow \mathbb{Z}$ be the rounding function sending x to the nearest integer. By problem 2.12 (or an easy variant of it), every subset of $\mathbb{Z}$ is open, so f is open. It is not continuous, however. In particular, it is not continuous at the point $\frac{1}{2} \in \mathbb{R}$.

b)Yes, we showed that f gives a bijection between the open sets of M and the open sets of N. In particular, f^{-1} is continuous and since f is a bijection, $f(U) = (f^{-1})^{pre}(U)$ is open.

c)Yes. Since f is a bijection, $(f^{-1})^{pre}(U) = f(U)$, so f^{-1} is continuous (since f is open). Thus f and f^{-1} are continuous bijections, so f is a homeomorphism.

d)No. Consider the map $f(x) = x(x - 1)(x + 1)$, and let $U = (0, 1)$ Then $f(U) = (0, \frac{1}{\sqrt{3}}]$, which is not open.

e)Yes! We just need to show that f is injective (1-to-1). Suppose not. Then we have an $a < b$ with $f(a) = f(b)$. Then f attains a maximum and minimum on the interval $[a, b]$ (we'll discuss this in detail when talking about the compactness theorem. Or: See Theorem 23 in Ch. 1). Let $U = (a, b)$. We'll see that $f(U)$ is not open. If both of these are $f(a)$, then f is constant in the interval and $f(U) = \{f(a)\}$ is not open. If at least one of the max/min is not $f(a)$, then we see that $f(U)$ is either a half-open or a closed interval, so $f(U)$ is not open.

f)Consider the map that send the point with polar coordinates $(1, \theta)$ to the point with polar coordinates $(1, 2\theta)$. This loops the circle around itself twice. This function is surjective, open, and continuous, but it is not injective (both $(1, 0)$ and $(1, \pi)$ are sent to $(1, 0) = (1, 2\pi)$)

26. 2.6

a) If $S \subseteq T$, then $\text{cl}(S) = \bigcap_{\text{closed } K \supseteq S} K \subseteq \bigcap_{\text{closed } K \supseteq T} K = \text{cl}(S)$, where the $\subseteq$ above holds since the intersection is over a smaller collection of sets K. OR Consider a limit point of S. Since S is a subset of T, it's also a limit point of T.

b) Either use part a and the fact that $\text{int}(S) = (\text{cl}(S^C))^C$ OR:

$\text{int}(S) = \bigcup_{\text{open } U \subseteq S} U \subseteq \bigcup_{\text{open } U \subseteq T} U = \text{int}(T)$, where the $\subseteq$ is because the union is over a larger collection of sets.

27. 2.81

a)We'll use the ε-δ definition of continuity to show that $d_M : M \times M \to \mathbb{R}$ is continuous with respect to the sum-metric on $M \times M$. Note that the function d_M is simply the distance on M. That is, d takes in a pair of points from M: (p, q) and outputs the distance between the two points in M: $d_M(p, q)$.

Fix $(p, q) \in M \times M$ and an $\varepsilon > 0$. We need to show that there is a $\delta > 0$ so that $d_{\text{sum}}((p, q), (p', q')) < \delta \to d_{\mathbb{R}}(d_M(p, q), d_M(p', q')) < \varepsilon$. We'll see that $\delta = \varepsilon$ works. Without loss of generality, $d_M(p, q) > d_M(p', q')$ (if it's not clear why, read the rest and fill in the other case). Note that $d_{\text{sum}}((p, q), (p', q')) = d_M(p, p') + d_M(q, q')$ and $d_{\mathbb{R}}(d_M(p, q), d_M(p', q')) = |d_M(p, q) - d_M(p', q')| = d_M(p, q) - d_M(p', q') \leq d_M(p, p') + d_M(p', q') + d_M(q', q) - d_M(p', q') \leq d_M(p, p') + d_M(q, q') < \delta$. Thus if $\delta = \varepsilon$, we see that the conclusion is $d_{\mathbb{R}}(d_M(p, q), d_M(p', q')) < \varepsilon$.

b)Yes! We showed (in class) that a function from $M \times N$ to any metric space is continuous in the sum-metric if and only if it is continuous in the Euclidian metric if and only if it is continuous in the max-matric.

28. 2.83

a)The difficult condition in each case is the triangle inequality, which is what I'll explain for each of the three below.

d_E: We need to show that for any $p = (p_1, p_2), q = (q_1, q_2)$, and $r = (r_1, r_2)$, $d_E(p, q) \leq d_E(p, r) + d_E(r, q)$. Let's consider what $(d_E(p, q))^2$ is (just so we don't have to think in terms of square roots yet): it's $d(p_1, q_1)^2 + d(p_2, q_2)^2$, i.e. it's just

$$|(d(p_1, q_1), d(p_2, q_2))|^2 \leq |(d(p_1, r_1) + d(r_1, q_1), d(p_2, r_2) + d(r_2, q_2))|$$

Let x be the vector $(d(p_1, r_1), d(p_2, r_2))$ and let y be the vector $(d(r_1, q_1), d(r_2, q_2))$. Then we need to bound $\langle x + y, x + y \rangle$:

$$\langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

where the inequality is due to the Cauchy-Schwartz inequality.

What this means is that $(d_E(p, q))^2 \leq (d_E(p, r) + d_E(r, q))^2$. Taking square roots of both sides yields the triangle inequality.

d_{sum}:$d_{\text{sum}}(p, q) = d_{\text{sum}}((p_1, p_2), (q_1, q_2)) = d(p_1, q_1) + d(p_2, q_2) \leq d(p_1, r_1) + d(r_1, q_1) + d(p_2, r_2) + d(r_2, q_2) = (d(p_1, r_1) + d(p_2, r_2)) + (d(r_1, q_1) + d(r_2, q_2)) = d_{\text{sum}}(p, r) + d_{\text{sum}}(r, q)$

$d_{\max}$: $d_{\max}(p, q) = d_{\max}((p_1, p_2), (q_1, q_2)) = \max\{d(p_1, q_1), d(p_2, q_2)\} \leq \max\{d(p_1, r_1) + d(r_1, q_1), d(p_2, r_2) + d(r_2, q_2)\} \leq \max\{d(p_1, r_1), d(p_2, r_2)\} + \max\{d(r_1, q_1), d(r_2, q_2)\} = d_{\max}(p, r) + d_{\max}(r, q)$

b) I am currently of the opinion that this part of the problem is false as stated. Extra points for anyone who can prove that.

29. 2.84

a)The difficult part is to show $\rho(p, q)$ satisfies the triangle inequality: Assuming $d(p, q) \geq d(p, r)$ and $d(p, q) \geq d(r, q)$:

$$\rho(p, q) = \frac{d(p, q)}{1 + d(p, q)} \leq \frac{d(p, r) + d(r, q)}{1 + d(p, q)} = \frac{d(p, r)}{1 + d(p, q)} + \frac{d(r, q)}{1 + d(p, q)} \leq \frac{d(p, r)}{1 + d(p, r)} + \frac{d(r, q)}{1 + d(r, q)} = \rho(p, r) + \rho(r, q$$

If $d(p,q) < d(p,r)$, then since $x \mapsto \frac{x}{1+x}$ is an increasing function:

$$\rho(p,q)\frac{d(p,q)}{1+d(p,q)} \leq \frac{d(p,r)}{1+d(p,r)} = \rho(p,r) \leq \rho(p,r) + \rho(r,q)$$

Similarly, if $d(p,q) < d(r,q)$.

b) The identity map is certainly a bijection. We need to see that it is continuous in both directions. One direction: Given $\varepsilon > 0$ and $p \in M$, we need to find $\delta > 0$ so that $d(p,q) < \delta \to \rho(p,q) < \varepsilon$. Let $\delta = \varepsilon$. Then since the function $x \mapsto \frac{x}{1+x}$ is an increasing function, $\rho(p,q) < \frac{\varepsilon}{1+\varepsilon} < \varepsilon$.

The other direction: Given $\varepsilon > 0$ and $p \in M$, we need to find $\delta > 0$ so that $\rho(p,q) < \delta \to d(p,q) < \varepsilon$.

If $\delta < \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$ and $\rho(p,q) < \delta$, then $\frac{d(p,q)}{1+d(p,q)} < \delta$, so $d(p,q)(1-\delta) < \delta$, and $d(p,q) < \frac{\delta}{1-\delta} \leq \frac{\frac{\varepsilon}{2}}{1-\frac{1}{2}} \leq \varepsilon$.

30. 2.92

a)

Claim 30.1. S is clopen if and only if ∂S is empty.

Proof. $\to$: Since S is closed, $\mathrm{cl}(S) = S$. Since S is open $\mathrm{int}(S) = S$. Thus $\partial S = \mathrm{cl}(S) \setminus \mathrm{int}(S) = S \setminus S = \emptyset$.

$\leftarrow$: $\partial S \mathrm{cl}(S) \setminus \mathrm{int}(S)$ is empty. Thus $\mathrm{cl}(S) = \mathrm{int}(S)$. But $\mathrm{int}(S) \subseteq S \subseteq \mathrm{cl}(S)$, so $S = \mathrm{cl}(S) = \mathrm{int}(S)$. Thus S is both closed (being equal to its closure) and open (being equal to its interior). □

b) $\partial S^C = \mathrm{cl}(S^C) \cap \mathrm{cl}((S^C)^C) = \mathrm{cl}(S^C) \cap \mathrm{cl}(S) = \mathrm{cl}(S) \cap \mathrm{cl}(S^C) = \partial S$.

c)

Claim 30.2. ∂S is closed for any set S.

Proof. $\partial S = \mathrm{cl}(S) \setminus \mathrm{int}(S) = \mathrm{cl}(S) \cap (\mathrm{int}(S))^C$. But this is the intersection of two closed sets, so is closed. □

Then since the closure of a closed set is itself, $\partial\partial S = \mathrm{cl}(\partial S) \setminus \mathrm{int}(\partial S) = \partial S \setminus \mathrm{int}(\partial S) \subseteq \partial S$. d)

Claim 30.3. If $A \subseteq B$, then $\mathrm{int}(A) \subseteq \mathrm{int}(B)$.

Proof. $\mathrm{int}(A) = \bigcup_{\text{open } U \subseteq A} U \subseteq \bigcup_{\text{open } U \subseteq B} U = \mathrm{int}(B)$ □

So, $\partial\partial\partial(S) = \mathrm{cl}(\partial\partial S) \setminus \mathrm{int}(\partial\partial S) = \partial\partial S \setminus \mathrm{int}(\partial\partial S) = (\mathrm{cl}(\partial S) \setminus \mathrm{int}(\partial S)) \setminus \mathrm{int}(\partial\partial S) = \mathrm{cl}(\partial S) \setminus (\mathrm{int}(\partial S) \cup \mathrm{int}(\partial\partial S))$. But $\partial\partial S \subseteq \partial S$ (by part c), so $\mathrm{int}(\partial\partial S) \subseteq \mathrm{int}(\partial S)$. Thus $\partial\partial\partial(S) = \mathrm{cl}(\partial S) \setminus (\mathrm{int}(\partial S) \cup \mathrm{int}(\partial\partial S)) = \mathrm{cl}(\partial S) \setminus \mathrm{int}(\partial S) = \partial\partial S$.

e)

Claim 30.4. $\mathrm{cl}(S \cup T) \subseteq \mathrm{cl}(S) \cup \mathrm{cl}(T)$ and $\mathrm{int}(S) \cup \mathrm{int}(T) \subseteq \mathrm{int}(S \cup T)$.

Proof. Suppose $(p_n)_{n \in \mathbb{N}}$ is a sequence in $S \cup T$ which converges to the point p. Then either infinitely many of the p_n are in S or infinitely many are in T. Thus we can take a subsequence entirely in S or entirely in T. Thus p is either a limit point of S or a limit point of T. Thus $\lim(S \cup T) \subseteq \lim(S) \cup \lim(T)$, so $\mathrm{cl}(S \cup T) \subseteq \mathrm{cl}(S) \cup \mathrm{cl}(T)$.

Every open subset of S or T is an open subset of $S \cup T$. Thus $\mathrm{int}(S) \cup \mathrm{int}(T) \subseteq \mathrm{int}(S \cup T)$. □

So, $\partial(S \cup T) = \mathrm{cl}(S \cup T) \setminus \mathrm{int}(S \cup T) \subseteq \mathrm{cl}(S) \cup \mathrm{cl}(T) \setminus \mathrm{int}(S \cup T) \subseteq \mathrm{cl}(S) \cup \mathrm{cl}(T) \setminus (\mathrm{int}(S) \cup \mathrm{int}(T)) \subseteq \partial S \cup \partial T$.

f)$\mathbb{Q} \subseteq \mathbb{R}$. The boundary is all of $\mathbb{R}$, and the boundary of that is $\emptyset$.

g) Consider $S = \{0\}$ and $T = \mathbb{R}$ both as subsets of $\mathbb{R}$. $\partial S = S = \{0\}$ and $\partial T = \emptyset$. So $\partial(S \cup T) = \emptyset \subsetneq \{0\} = \partial S \cup \partial T$.

31. 2.40

a) Suppose $((p_n, f(p_n))$ is a sequence in $M \times \mathbb{R}$ which converges in $M \times \mathbb{R}$, say to (p, r). Then the sequence (p_n) converges to p in M and the sequence $(f(p_n))$ converges to r in $\mathbb{R}$. By continuity of f, $f(p) = r$. But that means (p, r) is in the graph of f.

b) Consider the set fM. It's a compact subset of $\mathbb{R}$. Thus $M \times fM$ is compact. But the graph of f is a closed (by a) subset of $M \times fM$. A closed subset of a compact set is compact (Theorem 35), so the graph of f is compact.

c) We'll use the convergent sequence condition for continuity. Suppose (p_n) converges in M to p. We need to show that $(f(p_n))$ converges to $f(p)$ in $\mathbb{R}$. Suppose towards a contradiction that it doesn't. Then there is some $\varepsilon > 0$ so that infinitely many $f(p_k)$'s are not in $B_\varepsilon(f(p))$. Look at the subsequence (q_k) consisting of those (p_n) where $d(f(p_n), f(p)) > \varepsilon$. Then the sequence $(q_k, f(q_k))$ is a sequence in the graph of f. That sequence has a convergent subsequence in the graph of f: $(r_l, f(r_l))$. But (r_l) converges to p, so the limit of this sequence is (p, s) for some s. On one hand, this s cannot be $f(p)$ since $f(r_l)$ is never within ε of $f(p)$. On the other hand, this s must be $f(p)$ since (p, s) is in the graph of f. This is the contradiction we wanted.

d)$f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

32. 2.41

S^2 is compact, since it's a closed and bounded subset of $\mathbb{R}^3$. $\mathbb{R}^2$ is not compact since it's not a bounded set. Therefore, the two are not homeomorphic.

33. 2.43

The diameter of each of the K_n's is $\geq \mu$. So, for each n, there is a pair of points (x_n, y_n) in K_n so that $d(x_n, y_n) > \mu - \frac{1}{n}$ (note by problem 71 below, we really could take $d(x_n, y_n) \geq \mu$). Take a subsequence (p_n, q_n) of the sequence (x_n, y_n) which converges to (p, q) (we just used that $K_1 \times K_1$ is compact). We need to check two things. First, the limit (p, q) is in $\cap_n K_n$. This is exactly the same as the proof that $\cap_n K_n$ is non-empty. Namely, after the first k entries, the sequence is entirely in $K_k \times K_k$. Therefore, since $K_k \times K_k$ is closed (since it's compact), the limit is in K_k. Since this is true for every k, the limit (p, q) is in $\cap_n K_n$. Second, we need to see that $d(p, q) \geq \mu$. This is by the triangle inequality and the ε-principle. We'll show that for each $\varepsilon > 0$, $d(p, q) > \mu - \varepsilon$. Given $\varepsilon > 0$, let k be large enough that $d(p_k, p) < \frac{\varepsilon}{3}$, $d(q_k, q) < \frac{\varepsilon}{3}$ and $\varepsilon > \frac{3}{k}$. Then $\mu - \frac{\varepsilon}{3} \leq \mu - \frac{1}{k} \leq d(p_k, q_k) \leq d(p_k, p) + d(p, q) + d(q, q_k) < \frac{\varepsilon}{3} + d(p, q) + \frac{\varepsilon}{3}$. So $\mu < \varepsilon + d(p, q)$, or $d(p, q) > \mu - \varepsilon$.

34. 2.71

First let's note that $A \times B$ is compact. Define a function $f : A \times B \to \mathbb{R}$ by $f(a, b) = d(a, b)$. Once we show that this function is continuous, we will know that it attains its minimum on $A \times B$. That is, there will be points $a_0 \in A$ and $b_0 \in B$ so that $d(a_0, b_0)$ is as small as possible.

So, now let's show f is continuous. We can use any of the three product metrics on $A \times B$, so let's choose to use $d_{\max}$. Given $\varepsilon > 0$, we want to find a $\delta > 0$ so that $d_{\max}((a, b),(a', b')) < \delta \to |f(a, b) - f(a', b')| < \varepsilon$. Let $\delta = \frac{\varepsilon}{2}$. Then $d_{\max}((a, b),(a', b')) < \delta$ means $d(a, a') < \delta$ and $d(b, b') < \delta$. Then $d(a, b) \leq d(a, a') + d(a', b') + d(b', b) < 2\delta + d(a', b')$. Thus $d(a, b) - d(a', b') < 2\delta = \varepsilon$. By symmetry (same argument flipping a with a' and b with b'), $d(a', b') - d(a, b) < \varepsilon$, so $|d(a', b') - d(a, b)| < \varepsilon$.

35. Prelim 5

First of all, we'll use the fact that f isn't just continuous but is also uniformly continuous on $[0, 1] \times [0, 1]$. We'll show the same about g. Given $\varepsilon > 0$, we want a $\delta > 0$ so that $|x - y| < \delta \to |g(x) - g(y)| < \varepsilon$. So, let $\varepsilon > 0$ be given. There is some δ so that if $d_{\max}((p, q),(p', q')) < \delta$, then $|f(p, q) - f(p', q')| < \varepsilon$ (this is uniform continuity of f). We'll use this δ and show that if $|x - y| < \delta$, then $g(y) > g(x) - \varepsilon$. By symmetry, this implies that $|g(x) - g(y)| < \varepsilon$.

Given $|x - y| < \delta$, take the point $r \in [0, 1]$ so that $f(x, -)$ attains it maximum value at r (i.e., $f(x, r) = g(x)$). Then $d_{\max}((x, r),(y, r)) < \delta$. So $|f(x, r) - f(y, r)| < \varepsilon$. In particular, $g(y) \geq f(y, r) > f(x, r) - \varepsilon = g(x) - \varepsilon$. By the last paragraph, this is all we needed.

36. Prelim 13

a) Suppose towards a contradiction that the sequence does not converge to x. That means that for some $\varepsilon > 0$ there are infinitely many x_n so that $d(x_n, x) > \varepsilon$. Look at the subsequence of all x_n with $d(x_n, x) > \varepsilon$. There is a convergent subsequence of this, by compactness of A. But it's limit cannot be x. This is a contradiction.

b)$(0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots)$

37. Prelim 14

Let's show first that there are A and B so that $|f(x)| \leq A+Bx$ for all positive x. This is really enough, since then (by considering $f(-x)$) we know there are A', B' so that $|f(x)| \leq A' + B'(-x)$ for all negative x. By taking C to be the max of A and A' and D to be the max of B and B', we have $|f(x)| \leq C + D|x|$ for all $x \in \mathbb{R}$.

By uniform continuity, there is some δ so that $|x - y| \leq \delta$ implies $|f(x) - f(y)| < 1$. Let $A = |f(0)| + 1$ and $B = \frac{1}{\delta}$. Let's see that this works. Consider all x in the interval $[0, \delta]$. By the condition, $|f(x) - f(0)| < 1$, so $|f(x)| < |f(0)| + 1 = A < A + Bx$. Consider all x in the interval $[\delta, 2\delta]$. By the condition $|f(x) - f(\delta)| < 1$, so $|f(x)| < |f(\delta| + 1 < A + 1 = A + B\delta \leq A + Bx$. This should give a convincing picture; now let's make it formal.

By induction, let's show $|f(k\delta)| \leq A + k - 1$. It's true for 0 by definition of A as $|f(0)| + 1$. Suppose it's true for k. Then $|f((k + 1)\delta) - f(k\delta)| < 1$, so $|f((k + 1)\delta)| \leq |f(k\delta)| + 1 \leq A + k - 1 + 1 = A + (k + 1) - 1$. Now, given any positive x, there is some k so that $k\delta \leq x < (k+1)\delta$. Then $|f(x) - f(k\delta)| < 1$, so $|f(x)| < |f(k\delta)| + 1 \leq A + k - 1 + 1 = A + k = A + B \cdot k\delta \leq A + Bx$. By the first paragraph, we needed only to worry about positive x, so we're done.

38. 2.54

No. Consider two closed circles in $\mathbb{R}^2$ which intersect at a point on their bounderies. The interior is two open circles, which is not connected.

39. 2.55

No to both. Consider $\mathbb{Q}$ as a subset of $\mathbb{R}$. The closure is $\mathbb{R}$ which is connected, and the interior is $\emptyset$, which is connected.

40. 2.58

Let a be any point in U. Consider the set S consisting of all points b so that there is some path in U from a to b. S is non-empty as $a \in S$ (the trivial path connecting a to a). We'll show that it's clopen in U. Given a point $b \in S$, there is some $\varepsilon > 0$ so that $B_\varepsilon(b) \subseteq U$. Then there is a path in U from b to any point c in $B_\varepsilon(b)$. By concatenating paths, this gives a path from a to any point c in $B_\varepsilon(b)$ (first go from a to b, then go from b to the point c). Thus $B_\varepsilon(b) \subseteq S$, showing that S is open. To see that S is closed in U, we'll check that $U \setminus S$ is also open. Suppose $b \in U \setminus S$. Then there is some $\varepsilon > 0$ so that $B_\varepsilon(b) \subseteq U$. For any $c \in B_\varepsilon(b) \subseteq U$, there is a path from b to c. Thus there is a path from c to b. Thus, if there were a path from a to c, there would also be a path from a to b. So there is no path from a to c. We conclude that $B_\varepsilon(b) \subseteq U \setminus S$, so $U \setminus S$ is open. Thus S is a clopen non-empty subset of U. So S must be all of U, since U is connected. Thus for any point b in U there is a path from a to b. Finally, given any two points b and c in U, there is a path from b to a and a path from a to c, so there is a path from b to c. Thus U is path connected.

41. 2.59

In fact, the annulus is path-connected. Given two points in polar coordinates (θ, x) and (θ_2, y), there is a path going from (θ, x) to (θ, y) (moving directly, not changing θ throughout the path). There is also a path from (θ, y) to (θ_2, y) moving around the annulus keeping y fixed. Concatenating these paths gives us a path from one point to the other. This shows path-connectedness, thus connectedness.

42. 2.68

Suppose, towards a contradiction, that $X = (a, b)$ and $Y = [a, b)$ are homeomorphic. Then $Y \setminus \{a\}$ is homeomorphic to $X \setminus \{c\}$ for some point $c \in (a, b)$. But $X \setminus \{c\} = (a, c) \cup (c, b)$, which is disconnected and $Y \setminus \{a\} = (a, b)$, which is connected. This is a contradiction, showing that X and Y are not homeomorphic.

43. 2.78

a)Let M be a connected metric space, $\varepsilon > 0$ and let a, b be two points in M. We want to show that there is an ε-chain taking a to b.

Let $A = \{p \in M \mid \text{there is an } \varepsilon\text{-chain from } a \text{ to } p\}$. Then A is open. This is because if $p \in A$, then $B_\varepsilon(p) \subseteq A$. Also, A is closed: If (p_n) is a sequence in A limiting to p, then for some p_k, $d(p_k, p) < \varepsilon$. So, there is an ε-chain going from a to p_k, which gives an ε-chain from a to p. Thus A is clopen and $a \in A$, so it is non-empty. Therefore, $A = M$, and in particular, $b \in A$.

b)Suppose towards a contradiction that M is not connected. Then there are two closed non-empty disjoint sets A and B so that $M = A \cup B$. Then A and B are also both compact. Thus they attain a minimum distance to each other. That is, there is $a \in A$ and $b \in B$ so that whenever $c \in A$ and $e \in B$, $d(a, b) < d(c, e)$. Let ε be less than this distance. Now let's see, for our contradiction, that M is not ε-chain connected. That is, if you start in A, there is no way to get to B via an ε-chain. This is simply because inductively, $d(p_i, p_{i+1}) < \varepsilon$, so if $p_i \in A$, then p_{i+1} cannot be in B. Therefore, p_{i+1} must also be in A.

c)Yes. For any ε and any $a < b \in \mathbb{R} \setminus \mathbb{Z}$, you can get from a to b taking steps of size $< \varepsilon$ as follows: Let's define the p_i inductively. $p_0 = a$. Given $p_n < b$, we choose p_{n+1} as follows. If $d(p_n, b) < \varepsilon$, then $p_n = b$. Otherwise, if $p_n + \frac{\varepsilon}{2} \in \mathbb{R} \setminus \mathbb{Z}$, let $p_{n+1} = p_n + \frac{\varepsilon}{2}$. Otherwise, let $p_{n+1} = p_n + \frac{\varepsilon}{2} + \min\{\frac{\varepsilon}{4}, \frac{1}{2}\}$. In any of the three cases, $p_{n+1} \in \mathbb{R} \setminus \mathbb{Z}$ and the induction continues. But the distance from p_n to b goes down by at least $\frac{\varepsilon}{2}$ at each step, so eventually the distance is $< \varepsilon$, and then $p_{n+1} = b$.

d)No! Let M be the following subset of $\mathbb{R}^2$: $\{(x, y) \in \mathbb{R}^2 \mid y > \frac{1}{|x|}\}$.

44. Prelim 15

Solution 1: We need to find what $g(1)$ should be. For any $\varepsilon > 0$, there is some $\delta > 0$ so that $|x - y| < \delta$ implies $|h(x) - h(y)| < \varepsilon$. Let's look at what this means if we take $x \in (1 - \delta, 1)$. Then any point $y \in (1 - \delta, 1)$ is within δ of x. Thus for any point $y \in (1 - \delta, 1)$, $h(y) \in (h(x) - \varepsilon, h(x) + \varepsilon)$. Let's call the set $[f(x) - \varepsilon, f(x) + \varepsilon]$ by the name S_ε.

Let's use this over and over again to zoom in on the right value. For each n, let's do that above for $\varepsilon = \frac{1}{n}$. Then for some δ, whenever $y \in (1 - \delta, 1)$ then $h(y) \in S_\varepsilon$ and the diameter of $S_\varepsilon = 2\varepsilon = \frac{2}{n}$. Then there is a single number in $\cap_n S_{\frac{1}{n}}$. We define $g(1) =$ this single number.

Let's check that g is continuous: Given $\varepsilon > 0$, there is an N so that $\frac{1}{N} < \frac{\varepsilon}{2}$. But then we saw that there is some δ so that for every $y \in (1 - \delta, 1)$, $g(y) \in S_{\frac{1}{N}}$. By construction, $g(1)$ is also in $S_{\frac{1}{N}}$, but the diameter of $S_{\frac{1}{N}}$ is $\frac{2}{N} \leq \varepsilon$. Thus $|g(y) - g(1)| < \varepsilon$ whenever $y \in (1 - \delta, 1)$.

To see that there is only one choice of $g(1)$ which makes g continuous: Suppose there were two such functions g and g'. By continuity of both, $(g(1 - \frac{1}{n}))_n$ converges to $g(1)$ and $(g'(1 - \frac{1}{n}))_n$ converges to $g'(1)$. But these two sequences are the same, so $g(1) = g'(1)$ and $g = g'$.

Solution 2: (Credit to Tejas)

Lemma 44.1. *If $f : X \to Y$ is uniformly continuous, and (p_n) is a Cauchy sequence in X, then $(f(p_n))$ is a Cauchy sequence in Y.*

Proof. Let $\varepsilon > 0$ be given. We need to show that for some N, $n, m > N \to d(f(p_n), f(p_m)) < \varepsilon$. By uniform continuity, there is a $\delta > 0$ so that $d(p, q) < \delta \to d(f(p), f(q)) < \varepsilon$. Also, since (p_n) is Cauchy, there is some N so that $n, m > N \to d(p_n, p_m) < \delta$, so $n, m > N \to d(f(p_n), f(p_m)) < \varepsilon$. □

It follows that any sequence (p_n) in $[0, 1)$ which converges to 1 is Cauchy, so $(f(p_n))$ is Cauchy (and in $\mathbb{R}$) so convergent. We want to show that if (p_n) and (q_n) are two sequences in $[0, 1)$ which converge to 1, then $(f(p_n))$ and $(f(q_n))$ converge to the same limit. To see this, consider the sequence $(p_1, q_1, p_2, q_2, \ldots)$. This sequence also converges to 1, so is Cauchy, and so $(f(p_1), f(q_1), f(p_2), f(q_2), \ldots)$ is a convergent sequence. Let L be the limit. Then $(f(p_n))$ is a subsequence, so it has the same limit L. Also $(f(q_n))$ is a subsequence so it has the same limit L. Thus we see that there is a value L so that: For every sequence (p_n) in $[0, 1)$ which converges to 1, $(f(p_n))$ converges to L. Let $g(1) = L$, and the convergent sequence condition of continuity guarantees that g is continuous. Uniqueness follows, since any other function sends 1 to a value which is not L, so the convergent sequence condition for continuity guarantees that such a funciton is not continuous.

45. If $A \subseteq M$ is covering compact, then A is closed.

We'll show that if $x \notin A$ then $x \notin \mathrm{cl}(A)$. This is equivalent to $\mathrm{cl}(A) \subseteq A$, or that A is closed. Given $x \notin A$, define the following collection of open sets $U_n = \{y \in M \mid d(y, x) > \frac{1}{n}\}$. The collection of these $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is an open cover of A (since x is the only non-covered point, and $x \notin A$). Thus there is a finite subcover of $\mathcal{U}$. That means that there is some U_N so that $A \subseteq U_N$. But that means $d(x, a) > \frac{1}{N}$ for every $a \in A$. Thus $x \notin \mathrm{cl}(A)$.

46. 2.46

Let $\mathcal{U}$ be an open covering of $[a, b]$. We want to show that there is a finite subcover. We define the set $C = \{x \in [a, b] \mid [a, b] \text{ is covered by a finite subcover of } \mathcal{U}\}$. If we can show that $b \in C$, then we will know that $[a, b]$ can be covered by a finite subfamily of $\mathcal{U}$. The first step will be to show that b is the least upper bound for C. We know that C has a least upper bound.

Suppose towards a contradiction that it is c and $c < b$. There is a set $U \in \mathcal{U}$ which contains c. Since U is open, it contains $B_\varepsilon(c)$ for some $\varepsilon > 0$. Since c is the least upper bound for C, there is some $d \in C$ which is in $B_\varepsilon(c)$. So, $[a, d]$ is covered by some finite subcover $\mathcal{V}$ of $\mathcal{U}$. And thus $[a, c + \frac{\varepsilon}{2}]$ is covered by $\{U\} \cup \mathcal{V}$. So, $c + \frac{\varepsilon}{2}$ is in C contradicting c being an upper bound.

Thus b is the least upper bound for C. Now we'll show there is a finite subset of $\mathcal{U}$ which covers $[a, b]$. The argument is the same as above. There is a set $U \in \mathcal{U}$ which contains b. Since U is open, it contains $(b - \varepsilon, b]$ for some $\varepsilon > 0$. Since b is the least upper bound for C, there is some $d \in C$ which is in $(b - \varepsilon, b]$. So, $[a, d]$ is covered by some finite subcover $\mathcal{V}$ of $\mathcal{U}$. And thus $[a, b]$ is covered by $\{U\} \cup \mathcal{V}$. So, b is in C, and we are done.

47. 2.47

A is a closed subset of a covering compact set K. Let $\mathcal{U}$ be an open covering of A. Let $\mathcal{V}$ be $\mathcal{U} \cup \{M \setminus A\}$. Then $\mathcal{V}$ is an open cover of K. Thus there is a finite subcover $\mathcal{V}_0$ of K. If we exclude the set $K \setminus A$ from the cover $\mathcal{V}_0$, we have a finite subset of $\mathcal{U}$ which covers A. Thus A is covering compact.

48. 2.48

Suppose $\mathcal{U}$ is an open cover of fA. Let $\mathcal{V}$ be $\{f^{\text{pre}}(U) \mid U \in \mathcal{U}\}$. Then since f is continuous, $\mathcal{V}$ is a collection of open sets. Since $\mathcal{U}$ covered fA, $\mathcal{V}$ covers $f^{\text{pre}}(fA)$, which contains A, so $\mathcal{V}$ is an open cover of A. Thus there is a finite subset $\mathcal{V}_0$ of $\mathcal{V}$ which still covers A. It remains to show that the corresponding finite subset of $\mathcal{U}$ is a finite subcover of fA.

49. 2.49

If A is a closed subset of M, then by 2.47, it is covering compact, and by 2.48 fA is covering compact. By the lemma, fA is closed. So f takes closed subsets of M to closed subsets of N. Thus f^{-1} is continuous (the f^{-1} pre-image of a set A is just fA).

50. 2.50

Let $\mathcal{V}$ be the set of all sets $B_{\frac{\varepsilon}{2}}(p)$ where $p \in N$. Let $\mathcal{U}$ consist of the pre-images of sets in $\mathcal{V}$. That is, $\mathcal{U} = \{f^{\text{pre}}(B_{\frac{\varepsilon}{2}}(p)) \mid p \in N\}$. By the Lebesgue number lemma, there is a λ so that for any $q \in M$, there is some $U \in \mathcal{U}$ so that $B_\lambda(q) \subseteq U$. That means that $B_\lambda(q) \subseteq f^{\text{pre}}(B_{\frac{\varepsilon}{2}}(p))$ for some $p \in N$. But that means that for any two $a, b \in B_\lambda(q)$, $d(f(a), f(b) \leq d(f(a), p) + d(f(q), p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So, take $\delta = \lambda$, and we see that for each q, $d(x, q) < \delta$ implies $d(f(x), f(q)) < \varepsilon$.

51. 2.51

If $A_1 \supset A_2 \supset \ldots$ are non-empty covering compact sets, we want to show that $\bigcap_n A_n$ is non-empty. Suppose, towards a contradiction, that $\bigcap_n A_n = \emptyset$. Then we form the following cover of A_1: $U_k = \{M \setminus A_k\}$, and $\mathcal{U} = \{U_k \mid k \in \mathbb{N}\}$. Since the intersection $\bigcap_n A_n$ is empty (our assumption), $\mathcal{U}$ is an open cover of A_1. Thus there is a finite subcover of $\mathcal{U}$. But that means $A_1 \subseteq U_1 \cup U_2 \cup \ldots \cup U_k$ for some finite k. So $A_1 \subseteq U_k$. Thus, $A_k = \emptyset$, giving us a contradiction.

52. 2.52

Suppose $\mathcal{C} = \{C_i \mid i \in I\}$ (arbitrarily many of these) which have the finite intersection property (the intersection of any finitely many of them is non-empty). We want to show that the grand intersection $\bigcap_{C \in \mathcal{C}} C$ is non-empty. Suppose towards a contradiction that $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then let $\mathcal{U} = \{M \setminus C_i \mid i \in I\}$. This $\mathcal{U}$ is an open cover of M. Since M is covering compact, there is a finite subcover $\mathcal{U}_0 = \{M \setminus C_{i_1}, \ldots M \setminus C_{i_k}\}$. But $\mathcal{U}_0$ does not cover $C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}$. And this intersection is non-empty. This is a contradiction.

53. 2.53

We are given M so that every family of closed sets with the finite intersection property has a non-empty intersection. Given $\mathcal{U} = \{U_i \mid i \in I\}$ an open covering of M, we want to show that there is a finite subcover. Suppose towards a contradiction that there is no finite subcover. Then let $\mathcal{C}$ be the family of sets $C_i = M \setminus U_i$. If no finite collection of the U's covers M, then no finite intersection of the C's is empty. Thus $\mathcal{C}$ has the finite intersection property. Thus it has non-empty grand intersection. Say $x \in \bigcap_{i\in I} C_i$. Then x is not covered by $\mathcal{U}$. This is a contradiction to $\mathcal{U}$ being a covering of M.

54. 2.94

a)$\mathbb{Q}^n$ is a countable subset of $\mathbb{R}^n$. It is dense, since any open set contains a ball of radius ε for some ε, and any such ball contains an element of $\mathbb{Q}^n$.

b)We will construct a dense countable subset as follows. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{B_{\frac{1}{n}}(p) \mid p \in M\}$. Since M is compact, there is a finite subcover: $\{B_{\frac{1}{n}}(p_0), \dots B_{\frac{1}{n}}(p_k)\}$. Let $A_n = \{p_0, \dots, p_k\}$ and let $A = \cup_n A_n$. This A should be a countable dense subset of M. We need to show that for any a and any $\varepsilon > 0$, $B_\varepsilon(a) \cap A \neq \emptyset$. Let N be any number so that $\frac{1}{N} < \varepsilon$. Then q is in M, which is covered by $\{B_{\frac{1}{N}}(p_0), \dots B_{\frac{1}{N}}(p_k)\}$ for some points $p_0, \dots, p_k$ which we put in A. Thus for one of these points in A, $p_i \in B_\varepsilon(a) \cap A$.

55. 3.7

This is almost verbatim the same proof as on page 143-144. Mostly, it's changing b's to ∞'s.

56. 3.11

a) i) In this case, one of $\frac{f(\beta_n)-f(0)}{\beta_n}$ and $\frac{f(0)-f(\alpha_n)}{-\alpha_n}$ is $\leq \frac{f(\beta_n)-f(\alpha_n}{\beta_n-\alpha_n}$. Similarly, one of the two is $\geq \frac{f(\beta_n)-f(\alpha_n}{\beta_n-\alpha_n}$. This is because if the slope of the line from $(\alpha_n, f(\alpha_n))$ to $(0, f(0)$ and the slope of the line from $(0, f(0))$ to $(\beta_n, f(\beta_n))$ are both $> S$, then the slope from $(\alpha_n, f(\alpha_n))$ to $(\beta_n, f(\beta_n))$ is also $> S$. (You should probably write out something algebraic to verify this, but in fact the longer slope is a weighted average of the other two, so they can't be both bigger or both smaller than it). Thus since each of $\frac{f(\beta_n)-f(0)}{\beta_n}$ and $\frac{f(0)-f(\alpha_n)}{-\alpha_n}$ is approaching 0 as $n \to \infty$, it's also true that $\frac{f(\beta_n)-f(\alpha_n}{\beta_n-\alpha_n}$ must approach 0 as it is sandwiched between the two.

ii)A bit of a mess. I would recommend first writing out the case where $f'(0) = 0$ and $f(0) = 0$ then "shift and tilt" the proof.

iii)By the mean value theorem, $D_n = f'(c_n)$ for some $c_n \in (\alpha_n, \beta_n)$. Then $f'(c_n) \to f'(0)$ as $c_n \to 0$ (by continuity of f').

b) Let $\alpha_n = \frac{2}{(2n+1)\pi}$. Then $f'(\alpha_n) = 1$. Then for some small enough h_n, $\frac{f(\alpha_n+h)-f(\alpha_n)}{h} \geq 1 - \frac{1}{n}$. Let $\beta_n = \alpha_n + h_n$. Then D_n limits to 1.

57. 3.13

a) Suppose $f(0) = b > 0$ (the other case is symmetric). Consider the line l defined by $y = Lx + b$.

Claim 57.1. For any $x > 0$, the point $(x, f(x))$ is below the line l. i.e. $f(x) \leq Lx + b$.

Proof. Suppose towards a contradiction that $x > 0$ is such that $f(x) > Lx + b$. Then by the MVT, there is some point $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x}$. But then $f'(c) > L$ contradicting our assumption that $f'(c) < L$ for every c. □

Thus for any x so that $Lx + b < x$ (for any large x this holds), we see that $f(x) - x < 0$. Since $f(0) - 0 > 0$, it follows that $f(c) - c = 0$ for some c in between (by the IVT).

b)Consider the function $f(x) = x + e^{-x}$.

58. 3.14

a) This is clear at any point other than $x = 0$. At 0, we need to consider the following limit:

$$\lim_{h \to 0^+} \frac{e^{-\frac{1}{h}}}{h} = \lim_{h \to 0^+} \frac{\frac{1}{h}}{e^{\frac{1}{h}}}$$

,

which, by L'Hopital's rule is equivalent to

$$\lim_{h \to 0^+} \frac{-\frac{1}{h^2}}{-\frac{1}{h^2} e^{\frac{1}{h}}} = 0$$

.

Using a similar argument, you can show that $\lim_{h \to 0^+} \frac{p}{q} e^{-\frac{1}{x}} = 0$ for any polynomials p, q. Then by induction, you can see that each of the derivatives is defined and is 0 at 0.

b)No! If it were, then it would be the constant 0 function, but e is not.

c)First, β is zero outside of $(-1, 1)$, since either $e(1 - x)$ or $e(x + 1)$ is 0 outside of that interval. On $(-1, 1)$, it's easy to see that each term is > 0, so the product is > 0.

An induction on k shows that if f, g are C^k, then $f \circ g$ and fg are C^k (the proof is basically the product rule and chain rule used several times). Thus the composition and product of smooth functions here is smooth. Plugging in 0, you get $e^2 \cdot e^{-1} e^{-1} = 1$, so $\beta(0) = 1$.

d) Plug and chug.

59. 3.26

It should also state the assumption that for any ω_1, ω_2, either $\omega_1 \preceq \omega_2$ or $\omega_2 \preceq \omega_1$.

a)Suppose towards a contradiction that $lim_\Omega f = L_1$ and L_2. Let ε be $< \frac{|L_1 - L_2|}{2}$. Then there is an ω_1 and an ω_2 so that $\omega_1 \preceq \omega \to |f(\omega) - L_1| < \varepsilon$ and $\omega_2 \preceq \omega \to |f(\omega) - L_2| < \varepsilon$. Then using either $\omega_1 \preceq \omega_2$ or vice-versa, you get that $L_1 - L_2 < 2\varepsilon$, which contradicts the choice of ε.

b)The usual stuff: If $lim f = F$ and $lim g = G$, then let ω_f be good enough to ensure $f(\omega)$ is $\frac{\varepsilon}{2}$-close to F and let ω_g be good enough to ensure $g(\omega)$ is $\frac{\varepsilon}{2}$-close to G. Then whichever is bigger is good enough to ensure $f + g$ is is ε-close to $F + G$. etc.

c) Read the two definitions. They say exactly the same.

d)

$$\lim_\Omega R(f + cg) = \lim_\Omega (R(f) + cR(g)) =_{\text{by (b)}} \lim_\Omega R(f) + c \lim_\Omega R(g)$$

. The term on the right is $\int_a^b (f + cg)\, \mathrm{dx}$ and the term on the left is $\int_a^b f\, \mathrm{dx} + c \int_a^b g\, \mathrm{dx}$ by (c).

e)Replace $|f(\omega) - L| < \varepsilon$ by $d_M(f(\omega), L) < \varepsilon$.

60. 3.18

a) Suppose $(a_k)_{k\in\mathbb{N}}$ is a sequence in D_κ converging to a. We need to show that $osc_a(f) = \lim_{h\to 0}\sup\{f(x) \mid x \in (a-h, a+h)\} - \inf\{f(x) \mid x \in (a-h, a+h)\} \geq \kappa$. We'll show that for any h, $\sup\{f(x) \mid x \in (a-h, a+h)\} - \inf\{f(x) \mid x \in (a-h, a+h)\} \geq \kappa$, thus the limit is also $\geq \kappa$.

Given h, choose any a_k so that $|a - a_k| < \frac{h}{2}$. Since $osc_{a_k}(f) \geq \kappa$, $\sup\{f(x) \mid x \in (a_k - \frac{h}{2}, a + \frac{h}{2})\} - \inf\{f(x) \mid x \in (a - \frac{h}{2}, a + \frac{h}{2})\} \geq \kappa$. Thus there are points $b, c \in (a - \frac{h}{2}, a + \frac{h}{2}) \subseteq (a - h, a + h)$ so that $f(b) - f(c) \geq \kappa$. Thus $\sup\{f(x) \mid x \in (a-h, a+h)\} - \inf\{f(x) \mid x \in (a-h, a+h)\} \geq \kappa$.

b)$D = \bigcup_{n\in\mathbb{N}} D_{\frac{1}{n}}$, so D is a countable union of closed sets.

c)The complement of D is $\bigcap_{n\in\mathbb{N}}(D_{frac1n})^C$, so it's a countable intersection of open sets.

61. 3.28

a) If f is Riemann integrable, then $\lim_{\text{mesh}(P)\to 0} \to L$. Since the partitions described in the calculus book are a list of partitions with $mesh(P) \to 0$, it follows that the Riemann sums limit to L.

b)Consider $\chi_{\mathbb{Q}}$.

c)Really (if it were defined), we'd want the integral of $\chi_{\mathbb{Q}}$ to be 0, but the calculus-book style integral thinks it should be 1. So it's really no good for us.

62. 3.29

$1 \to 3$: For an interval (a_i, b_i), the diameter is just $b_i - a_i$.

$3 \to 2$: If $\cup S_i$ is a covering of Z and the total diameter is $< \varepsilon$. For each S_i, by definition of diameter $\sup(S_i) - \inf(S_i) = \text{diam}(S_i)$. Thus $\cup[\sup(S_i), \inf(S_i)]$ is a cover of Z satisfying 2.

$2 \to 1$: If $\cup[a_i, b_i]$ covers Z, then $\cup(a_i, b_i)$ covers $Z \setminus \cup_i\{a_i, b_i\}$. But then $Z \setminus \cup_i\{a_i, b_i\}$ is a zero-set and so is $\cup_i\{a_i, b_i\}$, since it is countable. Thus the union is a zero-set and contains Z, so Z is a zero-set.

63. 3.30

First show that $[0, 1]$ cannot be covered by finitely many open intervals (a_i, b_i) of total length < 1. Suppose towards a contradiction that it was covered by finitely many open intervals (a_i, b_i) of total length < 1. We can order these so that $a_1 < 0$, $a_2 < b_1$, $a_3 < b_2$,. . ., $a_n < b_{n-1}$, $b_n > 1$. This is because some interval must cover b_1, we just choose 2 to index this interval. Similarly, some interval must cover b_2, so we just choose 3 to index this interval, etc.

Now the total length is $\sum_{i\leq n}(b_i - a_i) = b_n - a_0 + \sum_{i\leq n-1} b_i - a_{i+1} \geq b_n - a_0 \geq 1$. Thus the total length couldn't have been < 1. So, we can't cover $[0, 1]$ with finitely many intervals of total length < 1.

Now, suppose towards a contradiction that $[0, 1]$ is covered by countably many open intervals of total length < 1. Then, by compactness of $[0, 1]$ a finite subset suffices to cover it. But the total length of this finite set of open intervals is < 1 contradicting what we just showed.

64. 3.31

Same as for normal cantor set.

65. 3.43

If f and g are antiderivatives of the same function, then $f'(x) = g'(x)$ for all x, thus $(f - g)'(x) = 0$ for all x, so $f - g$ is a constant function by an easy corollary of the MVT.

66. 3.47

$\frac{1}{M}$ is a bound for $\frac{1}{f(x)}$, and since $f(x)$ is never 0, $\frac{1}{f(x)}$ is continuous at exactly the points where f is continuous, so by the R-L theorem, $\frac{1}{f(x)} \in \mathcal{R}$.

If there is no bound M as in the problem, then $\frac{1}{f(x)}$ is unbounded, thus non-integrable.

67. 3.50

This is essentially exactly Corollary 29 in the text. Take a look at that.

68. 3.51

a) Corollary 26
b) Consider $f(x) = -1$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \in \mathbb{R} \setminus \mathbb{Q}$.
c) Corollary 26
d) Corollary 26
e) Consider f as in (b)
f) Corollary 26
g) Corollary 26 ($g(x) = \sqrt{x}$ for $x \geq 0$ and $g(x) = 0$ for $x < 0$ is a continuous function).