The following 9 problems are the E problems for 3 of the last 4 qualifying exams (the first 3 that I found the tex files for). You should be able to do all of the problems with material from Math 770, though many of them require some very clever solutions. Don't get discouraged.

Due Date: 12/16 (if on paper, because I leave town on 12/17) or 12/20 if in pdf.

E1. Let M model PA.

1. Show that there is no formula $\varphi(x, y)$ so that every subset $D \subseteq M$ definable (with parameters) is defined in M by $\varphi(x, b)$ for some b.

2. Show that for every $c \in M$, there is a formula $\varphi(x, y)$ so that every subset $D \subseteq [0, c)$ definable (with parameters) is defined in M by $\varphi(x, b)$ for some b.

E2. Ordinal addition and multiplication are not commutative. Generalize this as follows: Let $\star$ be any binary function on ω_1. Assume that $\alpha \star 2 > \alpha$ for all $\alpha > \omega$, and that $\star$ is continuous in the sense that $\alpha \star \beta = \sup_{n\in\omega}(\alpha \star \beta_n)$ whenever $\beta_0 < \beta_1 < \beta_2 < \cdots$ and $\beta = \sup_{n\in\omega} \beta_n$. Prove that $\star$ is not commutative.

E3. Call a model M "nice" iff for every $a, b \in M$, there is an automorphism of M that moves a to b. Let T be a theory in a countable language. Show that if T has a nice model of some infinite cardinality, then T has nice models of all infinite cardinalities.

E4. A countable graph $\mathcal{G} = \langle V, E\rangle$ is *the random graph* if for every pair of disjoint finite sets $A, B \subseteq V$, there is a vertex $v \in V$ such that $\mathcal{G}$ has edges between v and every element of A, but no edge between v and any element of B. Note that this determines a unique countable graph. Prove that the theory of the random graph is not finitely axiomatizable.

E5. Let δ be any ordinal and let $\gamma = \omega^\delta$ (under ordinal exponentiation). Let $\mathcal{U} = \{S \subseteq \gamma\colon S$ has order type ω and is unbounded in $\gamma\}$. Prove that $|\mathcal{U}|$ is 0 or $|\gamma|^{\aleph_0}$.

E6. Let T be a consistent r.e. axiomatizable extension of Peano Arithmetic. Show that there is an e such that the partial recursive function φ_e is total, but T does not prove that φ_e is total.

E7. Let L be a language which includes a unary relation symbol R. Let φ be an L-sentence and Γ a set of L-sentences neither of which contains the symbol R. If Γ proves φ in the language L, must there be a deduction of φ from Γ in which R does not occur (i.e., in the language $L - \{R\}$)? If so, prove that there is always such a deduction; and if not, describe Γ and φ which provide a counterexample.

E8. Show that there exists an $\mathcal{N} \models \mathrm{PA}$ and an $a \in \mathcal{N} \smallsetminus \mathbb{N}$ so that a is definable in $\mathcal{N}$.

E9. Let α, β and γ be ordinals. Prove that the six sums,

$$\begin{array}{ll} \alpha+\beta+\gamma, & \alpha+\gamma+\beta, \\ \beta+\alpha+\gamma, & \beta+\gamma+\alpha, \\ \gamma+\alpha+\beta, & \gamma+\beta+\alpha, \end{array}$$

cannot all be different.