Math 571: Mathematical Logic
4th homework set, due at 2:25pm on Monday, October 28th.

Bring your solutions to class, or slide them under the door of Evans 867.

1. Fix a language L, and let $\mathcal{A}$ be an L-structure, $\mathcal{B}$ be a substructure of $\mathcal{A}$, and let α be a quantifier-free L-formula. Show that if $\mathcal{A} \models \forall v_1 \forall v_2 \dots \forall v_k \alpha$, then $\mathcal{B} \models \forall v_1 \forall v_2 \dots \forall v_k \alpha$.

2. Fix a language L; recall that for a set Σ of L-sentences $Mod(\Sigma) := \{ L\text{-structures } \mathcal{A} \mid \mathcal{A} \models \Sigma\}$, and for a class K of L-structures $Th(K) := \{ L\text{-sentences } \theta \mid K \models \theta\}$. For two of the following, provide examples; for two others, prove that there are no examples.
   (a) Two sentences α and β such that $Mod(\alpha) \subsetneq Mod(\beta)$.
   (b) Two structures $\mathcal{A}$ and $\mathcal{B}$ such that $Th(\mathcal{A}) \subsetneq Th(\mathcal{B})$.
   (c) A class of structures K such that $Mod(Th(K)) = K$.
   (d) A class of structures K such that $Th(Mod(Th(K))) \subsetneq Th(K)$.

3. Fix a function $f : \mathbb{R} \to \mathbb{R}$. Consider a structure $(\hat{M}, \hat{+}, \hat{\cdot}, \hat{0}, \hat{1}, \hat{<}, \hat{=}, \hat{f})$ so that $(\mathbb{R}, +, \cdot, 0, 1, <, =, f) \prec (M, \hat{+}, \hat{\cdot}, \hat{0}, \hat{1}, \hat{<}, \hat{=}, \hat{f})$. (Prove one such exists).
   (a) Define the collection of positive infinitesimal numbers in M to be those i so that $M \models 0 < v_1 \wedge v_1 < v_2[s]$ for every s sending v_1 to i and v_2 to a positive real number. (Show these exist). Define i to be infinitesimal if $|i|$ is either 0 or a positive infinitesimal.
   (b) Show that for each real number a, f is continuous at a if and only if for all infinitesimal i, $f(a + i) - f(a)$ is infinitesimal.
   (c) Call a number b in M infinite if $M \models v_2 < v_1[s]$ for every s sending v_1 to b and v_2 to a real number (Prove that these exist). Show that $\lim_{x\to\infty} f(x) = L$ if and only if for every infinite b, $f(b) - L$ is infinitesimal.
   (d) Show that for every real number a, if f is differentiable at a, then for any infinitesimal $f'(a) - \frac{f(a+i)-f(a)}{i}$ is infinitesimal.

4. Let $(\hat{\mathbb{N}}, \hat{+}, \hat{\cdot}, \hat{0}, \hat{1}, \hat{<}, \hat{=})$ be so that $(\mathbb{N}, +, \cdot, 0, 1, <, =) \prec (\hat{\mathbb{N}}, \hat{+}, \hat{\cdot}, \hat{0}, \hat{1}, \hat{<}, \hat{=})$ (prove one exists). Let ϕ be any formula with one free variable v_1. Let S be the set of prime numbers p so that $\mathbb{N} \models \phi[s(v_1|p)]$ for any/every s. Show that there is an element b in $\hat{\mathbb{N}}$ so that for each prime number $p \in \mathbb{N}$, $\hat{\mathbb{N}} \models$ "p divides b" if and only if $p \in S$.

5. Prove (with all the details) that DLOW has quantifier elimination. Recall DLOW is the theory in the language $L = \{<, =\}$ which states that the model is a dense linear order without endpoints.