Please solve the following problems.

1. (a) Let $E$ be an elliptic curve over $\mathbb{Q}$ with $j(E) \neq 0$. Show that $E$ has a Weierstrass equation over $\overline{\mathbb{Q}}$ of the form

$$y^2 + axy + y = x^3 \quad (a \in \overline{\mathbb{Q}})$$

and that $(0,0)$ is a torsion point on $E$. What is its order?

(b) Let $n$ be a positive integer and consider the problem of finding all nonzero integers $u, v, w$ satisfying

$$(*) \quad u/v + v/w + w/u = n$$

Show that this amounts to finding $E_n(\mathbb{Q})$ for a certain elliptic curve $E_n$ so long as $n \neq 3$.

(c) Show that the torsion subgroup of $E_n(\mathbb{Q})$ always has order divisible by 3. Find an $n$ for which the torsion subgroup has order larger than 3.

(d) Show that the rank of $E_6(\mathbb{Q})$ is greater than 0. Find an upper bound for this rank. Find a non-torsion point in $E_6(\mathbb{Q})$ and the corresponding solution to $(*)$. Show that there are infinitely many solutions to $(*)$ with $n = 6$ and $u, v, w > 0$.

2. Suppose $p$ and $p-2$ are both primes ($p \geq 5$). Let $E$ be the elliptic curve over $\mathbb{Q}$ with equation $y^2 = x(x-2)(x-p)$.

(a) Using a computer algebra system such as MAGMA, investigate how the rank of $E(\mathbb{Q})$ varies with $p$. Give a conjecture, with cases depending on $p \pmod{8}$ as to what the rank is.

(b) Prove this conjecture if $p \equiv 7 \pmod{8}$. [Hint: show that many of the homogeneous spaces have no points by working in the reals or modulo powers of 2, $p$, or $p-2$.]

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