HOMEWORK 1, DUE FEB 15.

1. (a) Let $L/K$ be a finite Galois extension of local fields (both finite extensions of $\mathbb{Q}_p$). Let $x \in L$ have conjugates $x_1(= x), x_2, \ldots, x_n$ over $K$. Suppose $y \in L$ satisfies $|y - x| < |y - x_i|$ for $i \geq 2$. Show that $x \in K(y)$.

(b) Let $K$ be a finite extension of $\mathbb{Q}_p$ and $f \in K[X]$ be a separable, irreducible polynomial of degree $n$, defining extension $L$ of $K$ (i.e. $L \cong K[X]/(f)$). Show that every polynomial $h \in K[X]$ of degree $n$ that is close enough to $f$, is irreducible and that the extension $K[X]/(h)$ of $K$ is isomorphic to $L$.

2. (a) Construct an abelian extension of $\mathbb{Q}(\sqrt{2})$ that is not cyclotomic.

(b) Suppose $K/\mathbb{Q}$ is a quadratic extension. How are its discriminant and conductor related?

(c) Suppose $p$ is prime and $1 \pmod{3}$. Show that there is a unique Galois cubic extension of $\mathbb{Q}$ ramified only at $p$. What are its discriminant and conductor?

(d) Using the Jones-Roberts database, find the four Galois cubic fields of smallest discriminant that are not produced by the construction in part (c). What are their discriminants and conductors?

(e) There is a simple formula relating discriminant and conductor for all the fields in (b),(c),(d) above. Find two number fields for which this formula fails.