Analytic continuation in several complex variables

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Declaration

I hereby declare that the work in this thesis has been carried out by me in the Integrated Ph.D. program under the supervision of Professor Gautam Bharali, and in partial fulfillment of the requirements of the Master of Science degree of the Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship or any other title elsewhere.

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Abstract

We wish to study those domains in $\mathbb{C}^n$, for $n \geq 2$, the so-called domains of holomorphy, which are in some sense the maximal domains of existence of the holomorphic functions defined on them. We demonstrate that this study is radically different from that of domains in $\mathbb{C}$ by discussing some examples of special types of domains in $\mathbb{C}^n$, $n \geq 2$, such that every function holomorphic on them extends to strictly larger domains. Given a domain in $\mathbb{C}^n$, $n \geq 2$, we wish to construct the maximal domain of existence for the holomorphic functions defined on the given domain. This leads to Thullen’s construction of a domain (not necessarily in $\mathbb{C}^n$) spread over $\mathbb{C}^n$, the so-called envelope of holomorphy, which fulfills our criteria. Unfortunately this turns out to be a very abstract space, far from giving us a sense in general how a domain sitting in $\mathbb{C}^n$ can be constructed which is strictly larger than the given domain and such that all the holomorphic functions defined on the given domain extend to it. But with the help of this abstract approach we can give a characterization of the domains of holomorphy in $\mathbb{C}^n$, $n \geq 2$.

The aforementioned characterization is as follows: a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex. However, holomorphic convexity is a very difficult property to check. This calls for other (equivalent) criteria for a domain in $\mathbb{C}^n$, $n \geq 2$, to be a domain of holomorphy. We survey these criteria. The proof of the equivalence of several of these criteria are very technical – requiring methods coming from partial differential equations. We provide those proofs that rely on the first part of our survey: namely, on analytic continuation theorems.

If a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, is not a domain of holomorphy, we would still like to explicitly describe a domain strictly larger than $\Omega$ to which all functions holomorphic on $\Omega$ continue analytically. Aspects of Thullen’s approach are also useful in the quest to construct an explicit strictly larger domain in $\mathbb{C}^n$ with the property stated above. The tool used most often in such constructions is called “Kontinuitätssatz”. It has been invoked, without a clear statement, in many works on analytic continuation. The basic (unstated) principle that seems to be in use in these works appears to be a folk theorem. We provide a precise statement of this folk Kontinuitätssatz and give a proof of it.
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Chapter 1

Introduction and basic results

The initial part of this chapter will be devoted to understanding what it means for a function $f: \Omega \to \mathbb{C}$, where $\Omega$ is an open subset of $\mathbb{C}^n$, $n \geq 2$, to be holomorphic. For this purpose, let us introduce some convenient notations (which we shall follow in the later chapters also).

- We write $z \in \mathbb{C}^n$ as $z := (z_1, z_2, \ldots, z_n)$ and further, $z_j = x_j + iy_j$, $j = 1, 2, \ldots, n$.

- Given $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, and for any $z \in \mathbb{C}^n$, we set

  $$|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$ 

- We define the differential operators

  $$\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right),$$

  $$\frac{\partial}{\partial \overline{z}_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right),$$

  $$D^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$ 

- Given a point $a \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, the set

  $$D(a, r) := \{ z \in \mathbb{C}^n : |z_j - a_j| < r_j, \ j = 1, \ldots, n \}$$

  is called a \textit{polydisc} with centre at $a$ and \textit{polyradius} $r$. We will have the occasion where all the $r_j$'s are equal. In that case we will use the notation $D^n(a, r) := D(a, (r, \ldots, r))$, where $r > 0$.

Here are four plausible definitions for holomorphicity, each of which gives us the usual definition of holomorphicity when $n = 1$. Let $\Omega$ below be a domain (open connected set) in $\mathbb{C}^n$. 
**Definition 1.1.** A function \( f : \Omega \to \mathbb{C} \) is holomorphic if for each \( j = 1, \ldots, n \), and each fixed \( z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n \), the function
\[
\zeta \mapsto f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)
\]
is holomorphic in the classical one variable sense on the set
\[
\Omega(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) := \{ \zeta \in \mathbb{C} : (z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n) \in \Omega \}.
\]

**Definition 1.2.** A function \( f : \Omega \to \mathbb{C} \) that is continuously (real) differentiable with respect to each pair of variables \((x_i, y_i)\) separately on \( \Omega \) is said to be holomorphic if \( f \) satisfies the Cauchy-Riemann equations in each variable separately.

**Definition 1.3.** Let \( f : \Omega \to \mathbb{C} \) be continuous in each variable separately and locally bounded. The function \( f \) is said to be holomorphic if for each \( w \in \Omega \) there is an \( r = r(w) > 0 \) such that \( \overline{D^n(w, r)} \subset \Omega \) and
\[
f(z) = \frac{1}{(2\pi i)^n} \oint_{|z-w|=r} \cdots \oint_{|z_1-w_1|=r} \frac{f(z_1, \ldots, z_n)}{(z_1-z_1) \cdots (z_n-z_n)} \, d\zeta_1 \cdots d\zeta_n
\]
for all \( z \in D^n(w, r) \).

**Definition 1.4.** A function \( f : \Omega \to \mathbb{C} \) is holomorphic if for each \( a \in \Omega \) there is an \( r = r(a) > 0 \) such that \( \overline{D^n(a, r)} \subset \Omega \) and \( f \) can be written as an absolutely convergent power series
\[
f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z-a)^\alpha \quad \text{for all } z \in D^n(a, r),
\]
that converges uniformly on compact subsets of \( D^n(a, r) \).

It turns out that all four definitions are equivalent. To see this let us provisionally assume that \( f : \Omega \to \mathbb{C} \) is continuous. That 1.1 \( \Rightarrow \) 1.2 is trivial. Showing that 1.2 \( \Rightarrow \) 1.3 involves iterating the one variable Cauchy Integral Formula. To carry out this argument we must worry about the order of integration. Also we need to know, for instance that the functions
\[
\{ t \in \mathbb{C} : |t-w_k| = r \} \ni t \mapsto f(z^{(k)}, t, z_{k+1}, \ldots, z_n)
\]
are integrable for each fixed \( z^{(k)} \in m_{1 \leq j \leq k-1} \partial D^j(w_{j+1}, r) \) and \((z_{k+1}, \ldots, z_n) \in D^{n-k}((w_{k+1}, \ldots, w_n), r)\), \( k = 1, 2, \ldots, n-1 \). We do not have to worry about these points because we have taken \( f \) to be continuous on \( \Omega \). Now 1.3 \( \Rightarrow \) 1.4 using the same trick of expanding out a geometric series that we use in one complex variable. Finally, 1.4 \( \Rightarrow \) 1.1 because uniform convergence allows us to evaluate the \( \mathbb{C} \)-derivative with respect to a given \( z_j, j = 1, 2, \ldots, n \), term by term.

Just as Goursat showed that a univariate function admitting all first order partial derivatives (not necessarily continuous) and satisfying the Cauchy-Riemann equation is automatically \( \mathbb{C} \)-differentiable, it can be shown that the \textit{a priori} assumption that \( f \) is continuous is superfluous. This was shown by Hartogs. We direct the reader to [10][chapter 3] for a proof of Hartogs’ result.
We must mention that many fundamental results from the one-variable theory, e.g. the Cauchy Integral Formula, Liouville’s Theorem, the Open Mapping Theorem, Weierstrass’ Theorem, Montel’s Theorem, etc., have obvious generalizations to several variables. Their proofs, in most cases, involve subscripts by multi-indices in suitable places and iterating the one-variable argument, so we will skip these proofs. One theorem that has a slightly different form in higher dimensions is the principle of analytic continuation (which we shall use in proving the theorems presented below).

**Theorem 1.5** (Principle of analytic continuation). *Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( f \) be holomorphic on \( \Omega \). If \( f \) vanishes on a non-empty open subset of \( \Omega \) then \( f \equiv 0 \).*

The second part of this chapter focuses on how, in contrast to the above-named theorems, remarkably different the behaviour of holomorphic functions in multiple variables can be from the one-variable case. For instance, it is easy to describe domains with the property that every function holomorphic on them extend holomorphically to a strictly larger domain. It is less easy to describe completely the larger domain to which the above functions extend holomorphically. But we can provide a complete description for an important special class of domains. These are the motivations for the following results.

**Theorem 1.6.** Let \( V \) be a neighbourhood of \( \partial D^n(0, 1) \), \( n \geq 2 \) such that \( V \cap D^n(0, 1) \) is connected. Then, for any holomorphic function \( f \) on \( V \), there is a holomorphic function \( F \) on \( D^n(0, 1) \cup V \) so that \( F|_{V} \equiv f \).

**Proof.** Let \( \epsilon > 0 \) be such that if \( A := \{ z \in \mathbb{C}^n : 1 - \epsilon < |z_1| < 1, |z_j| < 1, j \geq 2 \} \cup \{ z \in \mathbb{C}^n : 1 - \epsilon < |z_2| < 1, |z_j| < 1, j \neq 2 \}, \)

\( A \subseteq V. \) Existence of such a \( \epsilon \) is guaranteed by the compactness of \( \partial D^n(0, 1) \). Let \( z' = (z_2, \ldots, z_n) \).

We note here that in this proof, and elsewhere in this report, if \( v = (v_1, \ldots, v_n) \) is a vector, then we shall, by a slight abuse of notation, denote \( |v| := \max_{j \leq n} |v_j| \) (to distinguish it from the Euclidean norm \( \| \cdot \| \)). If \( |z'| < 1 \), the function \( z_1 \mapsto f(z_1, z') \) is holomorphic on the annulus \( \{ z_1 \in \mathbb{C} : 1 - \epsilon < |z_1| < 1 \} \), so that

\[ f(z_1, z') = \sum_{\nu \in \mathbb{Z}} a_{\nu}(z') z_1^{\nu}, \]

for any \( z' \in D^{n-1}(0, 1) \). For each \( \nu \in \mathbb{Z} \)

\[ a_{\nu}(z') = \frac{1}{2\pi i} \int_{|z_1|=1} \frac{f(z_1, z')}{z_1^{\nu+1}} \, dz_1, \quad z' \in D^{n-1}(0, 1). \]

Differentiating under the integral sign, we get

\[ \frac{\partial a_{\nu}(z')}{\partial z_j} = \frac{1}{2\pi i} \int_{|z_1|=1} \frac{1}{z_1^{\nu+1}} \frac{\partial f}{\partial z_j}(z_1, z') \, dz_1 = 0, \quad j = 2, 3, \ldots, n, \]
for each $z' \in D^{n-1}(0, 1)$. Of course one must justify whether we can validly differentiate under the integral sign above. We shall give a criterion for being able to do this in the first part of Chapter 2. In view of Definition 1.2, $a_n(z')$ is holomorphic in $D^{n-1}(0, 1)$ for each $n \in \mathbb{Z}$. If now $1 - \varepsilon < |z_1| < 1$, $|z_2| < 1$, ..., $|z_n| < 1$, the function $z_1 \mapsto f(z_1, z')$ is holomorphic in the disc $|z_1| < 1$, so that its Laurent series contains no negative powers of $z_1$; i.e. $a_n(z') = 0$ for $n < 0$. As $D^{n-1}(0, 1)$ is connected, by the principle of analytic continuation, for $n < 0$, $z' \in D^{n-1}(0, 1)$. We define

$$F(z) := \begin{cases} f(z), & z \in V, \\ \sum_{n \in \mathbb{Z}} a_n(z')z_1^n, & z \in D^{n}(0, 1). \end{cases}$$

This latter series converges uniformly on compact subsets of $D^{n}(0, 1)$ and so is holomorphic there; further, it coincides with $f$ on $\{z \in \mathbb{C}^n : 1 - \varepsilon < |z_1| < 1, |z_j| < 1, j \geq 2\}$, a nonempty open subset of $V \cap D^{n}(0, 1)$ and so on the whole of $V \cap D^{n}(0, 1)$, as this set is open and connected. □

We need a definition in order to formulate our next theorem.

**Definition 1.7.** Let $\Omega$ be a domain in $\mathbb{C}^n$. We say that $\Omega$ is a Reinhardt domain if whenever $z = (z_1, z_2, \ldots, z_n) \in \Omega$ and $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$, we have $(e^{i\theta_1}z_1, e^{i\theta_2}z_2, \ldots, e^{i\theta_n}z_n) \in \Omega$.

**Theorem 1.8.** Let $\Omega$ be a Reinhardt domain in $\mathbb{C}^n$. Then for any holomorphic function $f$ on $\Omega$, there is a “Laurent series”

$$\sum_{\alpha \in \mathbb{Z}^n} a_{\alpha}z_{\alpha}$$

which converges uniformly to $f$ on compact subsets of $\Omega$. Moreover, the $a_{\alpha}$’s are uniquely determined by $f$.

**Proof.** We begin by proving the uniqueness. Let $w \in \Omega$ be a point with coordinates $(w_1, \ldots, w_n)$, $w_j \neq 0$ for all $j$. Then, since the series converges uniformly to $f$ on compact subsets of $\Omega$, we may set $z_j = w_j e^{i\theta_j}$, multiply by $e^{-i(\alpha_1\theta_1 + \alpha_2\theta_2 + \cdots + \alpha_n\theta_n)}$ and integrate term by term. Now we observe that

$$\int_{-\pi}^{\pi} e^{in\theta} d\theta = \begin{cases} 0, \quad &\text{if } n \neq 0, \\ 2\pi, \quad &\text{if } n = 0. \end{cases}$$

This gives, for $\alpha \in \mathbb{Z}^n$,

$$a_{\alpha} = w^{-\alpha}(2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(w_1 e^{i\theta_1}, w_2 e^{i\theta_2}, \ldots, w_n e^{i\theta_n}) e^{-i(\alpha_1\theta_1 + \alpha_2\theta_2 + \cdots + \alpha_n\theta_n)} d\theta_1 \cdots d\theta_n.$$ 

It seems that $a_{\alpha}$ depends on $w$, so that the coefficient $a_{\alpha}$ is not unique. So suppose $f(z) = \sum_{\alpha \in \mathbb{Z}^n} b_{\alpha}z_{\alpha}$ be a different representation of $f$ in $\Omega$. Then $0 = \sum_{\alpha \in \mathbb{Z}^n} (a_{\alpha} - b_{\alpha})z_{\alpha}, z \in \Omega$. Then, by the previous formula:

$$a_{\alpha} - b_{\alpha} = w^{-\alpha}(2\pi)^{-n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} 0 e^{-i(\alpha_1\theta_1 + \alpha_2\theta_2 + \cdots + \alpha_n\theta_n)} d\theta_1 \cdots d\theta_n = 0.$$

To prove the existence of an expansion as above, we first remark that if $D = \{z \in \mathbb{C}^n : r_j < |z_j| < R_j, -\infty < r_j < R_j, j = 1, 2, \ldots, n\}$ and $f$ is holomorphic on $D$, then, by iteration of the Laurent
expansion of one complex variable functions, it follows that \( f \) has an expansion in a Laurent series. Let \( w \in \Omega \). If \( \epsilon(w) > 0 \) is small enough, since \( \Omega \) is a Reinhardt domain, the set
\[
D(w, \epsilon(w)) = \{ z \in \mathbb{C}^n : |w_j| - \epsilon(w) < |z_j| < |w_j| + \epsilon(w) \}
\]
is contained in \( \Omega \). Since this is a set of the form \( D \) above, there is a Laurent expansion
\[
\sum_{\alpha \in \mathbb{Z}^n} a_\alpha(w)z^\alpha = f(z), \quad z \in D(w, \epsilon(w)),
\]
converges to \( f \) uniformly in a neighbourhood of \( w \). Now, if \( w_1 \in D(w, \epsilon(w)) \) and \( \sum_{\alpha \in \mathbb{Z}^n} a_\alpha(w_1)z^\alpha \) is the expansion corresponding to \( w_1 \) in a set \( D(w_1, \epsilon(w_1)) \subseteq \Omega \), then the uniqueness assertion above shows that \( a_\alpha(w) = a_\alpha(w_1) \). Hence the function \( w \mapsto a_\alpha(w) \) is locally constant on \( \Omega \) for any \( \alpha \in \mathbb{Z}^n \) and since \( \Omega \) is connected, \( a_\alpha(w) \equiv a_\alpha \), independent of \( w \). Hence any point \( z \in \Omega \) has a neighbourhood \( N(z) \subseteq \Omega \) so that the series
\[
\sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha
\]
converges uniformly to \( f(z) \) on \( N(z) \), hence uniformly to \( f(z) \) for \( z \) in any compact subset of \( \Omega \). \( \square \)

**Corollary 1.9.** Let \( \Omega \) be a Reinhardt domain such that for each \( j \), \( 1 \leq j \leq n \), there is a point \( z \in \Omega \) whose \( j \)-th coordinate is 0. Then any holomorphic function \( f \) on \( \Omega \) admits an expansion
\[
f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha
\]
which converges uniformly to \( f \) on compact subsets of \( \Omega \).

**Proof.** For \( n = 1 \) as the domain \( \Omega \) is a Reinhardt domain in \( \mathbb{C} \) containing 0, \( \Omega = D^1(0, r) \) for some \( r > 0 \) or \( \Omega = \mathbb{C} \). Therefore in the Laurent series expansion of \( f \) all the coefficients with negative index vanish. Hence we get the result. For \( n \geq 2 \), let \( w^1 \) represent an element in \( \Omega \) whose first coordinate is 0. So \( w^1 = (0, w_2, \ldots, w_n) \in \Omega \). As \( \Omega \) is a Reinhardt domain, there is a \( \epsilon > 0 \) such that \( D(w^1, \epsilon) \subseteq \Omega \), where
\[
D(w^1, \epsilon) := D^1(0, \epsilon) \times Ann(0, |w_2| - \epsilon, |w_2| + \epsilon) \times \cdots \times Ann(0, |w_n| - \epsilon, |w_n| + \epsilon).
\]
Fix \( z' = (z_2, z_3, \ldots, z_n) \in Ann(0, |w_2| - \epsilon, |w_2| + \epsilon) \times \cdots \times Ann(0, |w_n| - \epsilon, |w_n| + \epsilon) \). Then the function \( z_1 \mapsto f(z_1, z') \) is holomorphic on \( D^1(0, \epsilon) \). Using Theorem 1.8
\[
f(z) = f(z_1, z') = \sum_{\alpha_1 \in \mathbb{Z}} \left( \sum_{\alpha' \in \mathbb{Z}^{n-1}} a_{(\alpha_1, \alpha')} z'^{\alpha'} \right) z_1^{\alpha_1}, \quad z_1 \in D^1(0, \epsilon).
\]
But as \( z_1 \mapsto f(z_1, z') \) is holomorphic on \( D^1(0, \epsilon) \),
\[
\sum_{\alpha' \in \mathbb{Z}^{n-1}} a_{(\alpha_1, \alpha')} z'^{\alpha'} = 0, \quad \alpha_1 < 0 \text{ and } z' \in Ann(0, |w_2| - \epsilon, |w_2| + \epsilon) \times \cdots \times Ann(0, |w_n| - \epsilon, |w_n| + \epsilon).
\]
Ann(\{w_2| - \varepsilon, |w_2| + \varepsilon\} \times \cdots \times Ann(\{w_n| - \varepsilon, |w_n| + \varepsilon\}) is a Reinhardt domain in \(\mathbb{C}^{n-1}\). Hence from the uniqueness assertion in Theorem 1.8
\[ a_{(\alpha_1, \alpha')} = 0, \quad \text{for all } \alpha \in \mathbb{Z}^n : \alpha_1 < 0. \]

Similarly we can show that \(a_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} = 0\), if \(\alpha_j < 0\) for some \(j\). Hence \(f(z) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha\). \qed

**Corollary 1.10.** Let \(\Omega\) be a Reinhardt domain such that for each \(j, 1 \leq j \leq n\), there is a point \(z \in \Omega\) whose \(j\)-th coordinate is 0. Then any holomorphic function \(f\) on \(\Omega\) has an unique holomorphic extension \(F\) to the set \(\Omega' := \{(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n) : 0 \leq \lambda_j \leq 1, (z_1, z_2, \ldots, z_n) \in \Omega\}\).

**Proof.** By Corollary 1.9, \(f(z) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha\) for \(z \in \Omega\). Now let \(z = (\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n) \in \Omega'\), where for all \(j, \lambda_j \in [0, 1]\) and \((z_1, z_2, \ldots, z_n) \in \Omega\). As the series \(\sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha\) converges at the point \((z_1, z_2, \ldots, z_n)\), by Abel’s Lemma the series also converges at the point \(z = (\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n)\). Therefore
\[ F(z) := \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha, \quad z \in \Omega' \]
is a well defined holomorphic function on \(\Omega'\). Clearly \(F\) extends \(f\) since \(\Omega \subseteq \Omega'\). Now \(0 \in \Omega'\). Let \(z = (z_1, z_2, \ldots, z_n) \in \Omega'\). Then \(\lambda z = (\lambda z_1, \lambda z_2, \ldots, \lambda z_n) \in \Omega'\) for all \(\lambda \in [0, 1]\) by definition of \(\Omega'\). Therefore \(\Omega'\) is connected. Now \(\Omega\) is a nonempty open subset of \(\Omega'\). Hence by the principle of analytic continuation if there exists another holomorphic function \(G\) on \(\Omega'\) such that it also extends \(f\) then \(F(z) = G(z)\) for all \(z \in \Omega'\). \qed
Chapter 2

The Hartogs phenomenon

We saw in Chapter 1 that we can easily give examples of domains in \( \mathbb{C}^n \), \( n \geq 2 \), with the property that every function holomorphic on them extends holomorphically to a strictly larger domain. However, the proof of the above fact depended on the domains having a lot of symmetry. All the domains in Theorem 1.8 of Chapter 1, and in its corollaries, are Reinhardt. However, Hartogs showed that the phenomenon demonstrated by Theorem 1.6 holds true for arbitrary domains in \( \mathbb{C}^n \), \( n \geq 2 \). This will be demonstrated in Section 2.3 below. But first, we will need to go through a few technicalities.

2.1 Differentiating under the integral sign

Many theorems in Complex Analysis involve differentiating under the integral sign. However, this needs a justification. In many cases, the classical Leibniz Theorem does not suffice. The following proposition provides sufficient conditions that are general enough to be usable in many contexts.

**Proposition 2.1.** Let \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) be a measurable function that satisfies \( \lim_{x \to 0} |K(x)| = +\infty \), but which diverges sufficiently slowly at \( 0 \in \mathbb{R}^n \) that \( K \in L^1_{\text{loc}}(\mathbb{R}^n) \).

1. Suppose \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a continuous function having the following two properties:

   - There exists \( R > 0 \) such that, for each \( x \in \mathbb{R}^n \), \( f(x, y) = 0 \) for all \( y \in \mathbb{R}^n \setminus B(x, R) \);
   - The family \( \{ f(\cdot, y) : y \in \mathbb{R}^n \} \) is equicontinuous at each \( x \in \mathbb{R}^n \).

Then, the function

\[
g(x) := \int_{\mathbb{R}^n} K(y) f(x, y) \, dV(y) \tag{2.1}
\]

is well-defined and continuous.
2 Now suppose that, given $N \geq 1$, $\frac{\partial f}{\partial x^\alpha}$ exists for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq N$. Assume, furthermore, that the families $\left\{ \frac{\partial f}{\partial x^\alpha}(\cdot, y) : y \in \mathbb{R}^n \right\}$ are equicontinuous at each $x \in \mathbb{R}^n$, $|\alpha| \leq N$.

Then, $g \in C^N(\mathbb{R}^n)$, and we can, in fact, differentiate under integral sign.

**Proof.** Observe that, by hypothesis:

$$\int_{\mathbb{R}^n} K(y) f(x, y) \, dV(y) = \int_{\mathbb{R}^n} K(y) f(x, y) \, dV(y).$$

The integral on the right-hand side is convergent and hence $g$ is well defined.

Now note; for any $x_0 \in \mathbb{R}^n$:

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} |K(y)| |f(x, y) - f(x_0, y)| \, dV(y). \quad (2.2)$$

By the equicontinuity hypothesis, given $\epsilon > 0$, there exists $\delta_1 \equiv \delta_1(x_0, \epsilon) > 0$ such that $|x - x_0| < \delta_1$ implies $|f(x, y) - f(x_0, y)| < \epsilon$ for all $y \in \mathbb{R}^n$. Hence, by (2.2),

$$|g(x) - g(x_0)| \leq \int_{B(x_0, \rho + 1)} |K(y)| |f(x, y) - f(x_0, y)| \, dV(y) \leq \epsilon \|K\|_{L^1(\mathbb{R}^n)}.$$

whenever $|x - x_0| < \min(1, \delta_1(x_0, \epsilon))$.

This establishes continuity and hence part 1.

For clarity of explanation, we shall carry out the proof of part 2 in the following distinct steps:

**Step 1:** Illustrating a special case.

Pick a $j \leq n$. Then, from the first condition in part 1, we see that the analogue of that condition is true for $\frac{\partial f}{\partial x_j}$ also. Hence, the integral

$$I_j(x) := \int_{\mathbb{R}^n} K(y) \frac{\partial f}{\partial x_j}(x, y) \, dV(y) \quad (2.3)$$

is well defined. For any $t \in \mathbb{R} \setminus \{0\}$,

$$\left| \frac{g(x_0 + t e_j) - g(x_0)}{t} - I_j(x_0) \right| = \left| \int_{\mathbb{R}^n} K(y) \left\{ \frac{f(x_0 + t e_j, y) - f(x_0, y)}{t} - \frac{\partial f}{\partial x_j}(x_0, y) \right\} \, dV(y) \right|$$

$$= \left| \int_{\mathbb{R}^n} K(y) \left\{ \int_0^1 \frac{\partial f}{\partial x_j}(x_0 + t e_j, y) - \frac{\partial f}{\partial x_j}(x_0, y) \, ds \right\} \, dV(y) \right|. \quad (2.4)$$

Here $e_j$ denotes the $j$-th vector of the standard basis of $\mathbb{R}^n$. The last equation is obtained by applying the Fundamental Theorem of Calculus to the univariate functions

$$s \mapsto f(x_0 + s e_j, y).$$
Once again, by the equicontinuity hypothesis, given any $\epsilon > 0$ we can find a $\delta_2 \equiv \delta_2(x_0, \epsilon)$ such that $|t| < \delta_2 \implies \left| \frac{\partial f}{\partial x_j}(x_0 + ste_j, y) - \frac{\partial f}{\partial x_j}(x_0, y) \right| < \epsilon$ for all $y \in \mathbb{R}^n$ and for all $s \in [0, 1]$. Applying the above inequality to (2.4) gives us

$$\left| \frac{g(x_0 + te_j) - g(x_0)}{t} - I_j(x_0) \right| \leq \epsilon \|K\|_{L^1(B(x_0, R+1))},$$

whenever $0 < |t| < \min(1, \delta_2(x_0, \epsilon))$.

We have established that, for each $j \leq n$:

$$\frac{\partial g}{\partial x_j}(x_0) = \frac{\partial}{\partial x_j} \left[ \int_{\mathbb{R}^n} K(y) f(x_0, y) \, dV(y) \right] = \int_{\mathbb{R}^n} K(y) \frac{\partial f}{\partial x_j}(x_0, y) \, dV(y).$$

The continuity of $\frac{\partial g}{\partial x_j}$ follows by repeating the proof of part 1 with $\frac{\partial g}{\partial x_j}$ replacing $f$ in (2.1). Hence $g \in C^1(\mathbb{R}^n)$.

**Step 2:** Completing the proof.

We use induction to complete the proof. We are given $N \geq 1$. Assume we have the desired result for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$ for some $m < N$. Now we notice that step 1 was the base case of the induction. Given an $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m + 1$, there exists $\beta$ such that $|\beta| = m$ and a $j \leq n$ such that

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_{j-1}, \beta_j + 1, \beta_{j+1}, \ldots, \beta_n) =: (\beta, j).$$

We now repeat the argument in step 1 with the following replacements:

- $\frac{\partial f}{\partial \bar{z}^\beta}$ replacing $f$ in the relevant places;
- $\frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial \bar{z}^\beta} \right)$ replacing $\frac{\partial f}{\partial x_j}$ in the definition of $I_j$ in (2.3);

which establishes the result for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m + 1$. By induction, $g \in C^N(\mathbb{R}^n)$.

\[ \square \]

### 2.2 The $\overline{\partial}$-problem

Let $D$ be a domain in $\mathbb{C}^n$. For $n \geq 2$, the $\overline{\partial}$ -problem denotes the following system of PDE’s:

$$\frac{\partial u}{\partial \bar{z}_j} = f_j \quad \text{on } D \quad j = 1, \ldots, n, \quad (2.5)$$

subject to:

$$\frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j} \quad \text{for all } j \neq k, \quad (2.6)$$

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where \( f_1, \ldots, f_n \in L^1_{\text{loc}}(D) \). When \( n = 1 \), the compatibility condition (2.6) is vacuous, and the \( \bar{\partial} \)-problem is just the PDE (2.5). We note that if the \( f_j \)'s are not smooth, then (2.6) is interpreted in the sense of distributions. We must demand (2.6) when \( n \geq 2 \) because it is a necessary condition for (2.5) to have a smooth solution. To see this, first assume that the \( f_j \)'s are of class \( C^1 \). If a smooth solution \( u \) exists, then as it satisfies the PDE (2.5) we have for each \( j \), \( \frac{\partial u}{\partial z_j} = f_j \). As \( u \) is a smooth function, therefore all the mixed second-order partial derivatives of \( u \) exist and they are independent of the order of the variables. In particular as \( f_j \in C^1 \), (2.6) is satisfied. The same argument can be made in the weak sense if the \( f_j \)'s are not smooth.

Let us provide a solution to the \( \bar{\partial} \)-problem when \( n = 1 \), \( D = \mathbb{C} \) and the right-hand side is in \( C^k(\mathbb{C}) \) and has compact support. For this, we need an elementary fact from Advanced Calculus.

**Theorem 2.2 (The Generalized Cauchy Integral Formula).** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) such that \( \partial \Omega \) is a disjoint union of finitely many piecewise smooth simple closed curves. Let \( U \) be a neighbourhood of \( \overline{\Omega} \) and let \( f : U \to \mathbb{C} \) be of class \( C^1(U) \). Then for any \( w \in \Omega \),

\[
f(w) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - w} \, dz - \frac{1}{\pi} \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \frac{1}{z - w} \, dA(z).
\]

**Theorem 2.3.** Let \( k \geq 1 \) and let \( \phi \in C^k(\mathbb{C}) \), and suppose \( \phi \) is compactly supported. Then, the function

\[
u(z) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(w)}{w - z} \, dA(w)
\]

satisfies the PDE

\[
\frac{\partial u}{\partial \bar{z}} = \phi.
\]

Furthermore, \( u \in C^k(\mathbb{C}) \).

**Proof.** Since \( \phi \) is compactly supported, there exists \( r_0 > 0 \) such that \( \text{supp}(\phi) \subset \mathbb{B}(0; r_0) \). Now, given \( z \in \mathbb{C} \), let \( R > r_0 \) and such that \( z \in \mathbb{B}(0, R) \). By Theorem 2.2

\[
\phi(z) = -\frac{1}{\pi} \int_{\mathbb{B}(0, R)} \frac{1}{w - z} \frac{\partial \phi}{\partial \bar{z}}(w) \, dA(w) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w - z} \frac{\partial \phi}{\partial \bar{z}}(w) \, dA(w).
\] (2.7)

Now, from (2.7), and the fact that \( R \) can be arbitrarily large, the last integral represents \( \phi(z) \) for all \( z \in \mathbb{C} \). Now we observe that, by a change of variable

\[
u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w} \phi(w + z) \, dA(w).
\]

We observe that if we set

\[
K(w) := \frac{1}{w}, \quad w \in \mathbb{C} \setminus \{0\},
\]

\[
f(z, w) := \phi(w + z),
\]

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then the hypotheses of Proposition 2.1, are satisfied. Thus, \( u \in C^k(\mathbb{C}) \), and we can differentiate under the integral sign to get

\[
\frac{\partial u}{\partial z_k}(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w} \frac{\partial f}{\partial \overline{z}}(z, w) \, dA(w)
\]

\[
= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w-z} \frac{\partial \phi}{\partial \overline{z}}(w) \, dA(w) \quad \text{[change of variable]}
\]

\[
= \phi(z) \quad \text{for all } z \in \mathbb{C}. \quad \text{[by (2.7)]}
\]

The above plays a very important part in the solution of the \( \overline{\partial} \)-problem in \( \mathbb{C}^n \), \( n \geq 2 \), with compactly-supported data.

**Theorem 2.4.** Let \( f_j \in C^0_0(\mathbb{C}^n), j=1, \ldots, n \), where \( k > 0 \), and assume that (2.6) is fulfilled \( (n>1) \).

Then there is a function \( u \in C^0_0(\mathbb{C}^n) \) satisfying (2.5).

Now note that the part about \( u \) having compact support in this theorem is false for \( n=1 \) (take an arbitrary \( f_1 \in C_0^0 \) with Lebesgue integral different from 0).

**Proof.** We set

\[
u(z) := -\frac{1}{\pi} \int_{\mathbb{C}} f_1(w, z_2, \ldots, z_n) \frac{dA(w)}{w - z_1} = \frac{1}{\pi} \int_{\mathbb{C}} f_1(z_1 - w, z_2, \ldots, z_n) \frac{dA(w)}{w}.
\]

The second form of the definition shows that we can apply Proposition 2.1 to get that \( u \in C^k(\mathbb{C}^n) \). Now \( u(z) = 0 \) if \( |z_2| + \cdots + |z_n| \) is large enough. By Theorem 2.3 it follows that \( \frac{\partial u}{\partial z_1} = f_1 \). If \( k > 1 \), by differentiating under the sign of integration and using the fact that \( \frac{\partial f_1}{\partial z_1} = \frac{\partial f_k}{\partial z_1} \), we obtain

\[
\frac{\partial u}{\partial z_k}(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w - z_1} \frac{\partial f_k}{\partial \overline{z}_1}(w, z_2, \ldots, z_n) \, dA(w) = f_k(z).
\]

Hence \( u \) satisfies all the equations in (2.5), which means in particular \( u \) is analytic outside a compact set. As this complement is connected, and \( u(z) = 0 \) when \( |z_2| + \cdots + |z_n| \) is large, we deduce from the principle of analytic continuation, that \( u \) has compact support. \( \square \)

### 2.3 The Hartogs continuation theorem

Now we have sufficient background to prove a theorem that is nowadays called “Hartogs phenomenon.” Clearly, as mentioned in the beginning of Chapter 2, this theorem gives a generalization of Theorem 1.6.

**Notation:** \( H(\Omega) \) denotes the set of all holomorphic functions from \( \Omega \) into \( \mathbb{C} \).
Theorem 2.5. Let $\Omega$ be an open set in $\mathbb{C}^n$, $n > 1$, and let $K$ be a compact subset of $\Omega$ such that $\Omega \setminus K$ is connected. For every $u \in H(\Omega \setminus K)$ one can then find $U \in H(\Omega)$ so that $u = U$ in $\Omega \setminus K$.

Proof. Using the partition of unity we construct a function $\varphi \in C^\infty_0(\Omega)$, such that $\varphi$ is identically equal to 1 in a neighbourhood of $K$. Now we define

$$u_0(z) := \begin{cases} 0 & \text{if } z \in K, \\ (1 - \varphi(z))u(z) & \text{if } z \in \Omega \setminus K. \end{cases}$$

Then $u_0 \in C^\infty(\Omega)$, and we want to find $v \in C^\infty(\mathbb{C}^n)$, so that

$$U := u_0 - v$$

is holomorphic on $\Omega$. Now the $U$ will be analytic if $\bar{\partial}U = 0$, i.e.

$$\bar{\partial}v = \bar{\partial}u_0 = -u \bar{\partial}\varphi = f,$$

where $f$ is defined by

$$f(z) := \begin{cases} 0 & \text{if } z \in K, \\ -u(z) \bar{\partial}\varphi(z) & \text{if } z \in \Omega \setminus K, \\ 0 & \text{if } z \in \mathbb{C}^n \setminus \Omega. \end{cases}$$

Now we note that as $\varphi$ is compactly supported the components of $f$ are in $C^\infty_0(\mathbb{C}^n)$. Hence, if we write $f = \sum_{j=1}^n f_j \, d\overline{z}_j$, by (2.9) we have

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j} \quad \text{for all } j \neq k.$$

Therefore by Theorem 2.4, (2.8) has a $C^\infty_0(\mathbb{C}^n)$ solution. Furthermore, a careful look at the proof of Theorem 2.4 tells us that the solution given by the formula in Theorem 2.4 vanishes in the unbounded component of the complement of the support of $\varphi$. Now the boundary of this set belongs to $\Omega \setminus K$, so there exists an open set in $\Omega \setminus K$ where $v = 0$ and $u = u_0$. Hence the analytic function $U$ in $\Omega$ which coincides with $u$ on some open subset of $\Omega \setminus K$, and since this is a connected set, we have $u \equiv U$ on $\Omega \setminus K$. \qed
Chapter 3

Envelopes and domains of holomorphy

In view of the phenomenon seen in Chapters 1 and 2, we would now like to discuss, given a domain \( \Omega \), the “largest” domain, in the appropriate sense, to which all functions in \( H(\Omega) \) simultaneously extend. As our experience with the logarithm in 1-variable complex analysis shows, we need to take the help of germs to construct this “largest” domain. But because we will need to construct such objects for the simultaneous extension of a whole family of functions, we must introduce the idea of \( S \)-germs.

But first, we provide a definition.

**Definition 3.1.** Let \( X \) be a Hausdorff topological space. If there exists a map \( p : X \to \mathbb{C}^n \) such that \( p \) is a local homeomorphism then \( (X, p, \mathbb{C}^n) \) is called an (unramified) domain over \( \mathbb{C}^n \).

Let \( S \) be some fixed set. We want to define the sheaf \( O(S) \), the sheaf of \( S \)-germs of holomorphic functions on \( \mathbb{C}^n \). If \( U \) is a non-empty open set, then the pair \( (U, \{f_s\}_{s \in S}) \) will denote an arbitrary family of functions holomorphic on \( U \) indexed by \( S \). For \( a \in \mathbb{C}^n \) and \( (U, \{f_s\}_{s \in S}), (V, \{g_s\}_{s \in S}) \) with \( a \in U \cap V \), we say that these two pairs are equivalent if there exists a neighbourhood \( W \) of \( a \), \( W \subset U \cap V \) such that, for all \( s \in S \), \( f_s|_W \equiv g_s|_W \). An equivalence class with respect to this relation is called an \( S \)-germ of holomorphic functions at \( a \). We denote the set of all \( S \)-germs at \( a \) by \( O_a(S) \). We set \( O(S) := \bigsqcup_{a \in \mathbb{C}^n} O_a(S) \). We have a natural projection \( p = p_s : O_S \to \mathbb{C}^n \) defined by \( p(g_a) := a \) if \( g_a \in O_a(S) \). We define a topology on \( O(S) \) as follows: Let \( g_a \in O_a(S) \) and let \( (U, \{f_s\}_{s \in S}) \) be a representative of \( g_a \). Let \( [U, f_s : s \in S]_b \) denote the germ at \( b \) determined by the above pair for any \( b \in U \). Now we write

\[
N(U, \{f_s\}_{s \in S}) := \bigcup_{b \in U} [U, f_s : s \in S]_b.
\]

We topologize \( O(S) \) by demanding that the sets \( N(U, \{f_s\}_{s \in S}) \) form a fundamental system of neighbourhoods of \( g_a \). The proof of the following proposition is very routine and mechanical, so we shall skip its proof.

**Proposition 3.2.** The map \( p : O(S) \to \mathbb{C}^n \) is continuous and is a local homeomorphism of \( O(S) \) onto \( \mathbb{C}^n \). Further, \( O(S) \) is a Hausdorff topological space. The triple \( (O(S), p, \mathbb{C}^n) \) is an (unramified) domain over \( \mathbb{C}^n \).
3.1 Envelopes of holomorphy

Let \( p : X \to \mathbb{C}^n \) be a domain over \( \mathbb{C}^n \). A continuous function \( f : X \to \mathbb{C} \) is said to be holomorphic if for each open set \( U \subseteq X \) such that \( p|_U \) is a homeomorphism, \( f \circ (p|_U)^{-1} \in H(p(U)) \). We can extend the notion of holomorphicity to maps between domains in an analogous fashion. Now let \( p : X \to \mathbb{C}^n, p' : X' \to \mathbb{C}^n \) be domains over \( \mathbb{C}^n \). A continuous map \( u : X \to X' \) is called a local isomorphism if every point \( a \in X \) has a neighbourhood \( U \) such that \( u|_U \) is a homeomorphism onto an open set \( U' \subseteq X' \) and \( u|_U, (u|_U)^{-1} \) are holomorphic (on \( U \) and \( U' \) respectively). If, in addition, \( u \) is a homeomorphism of \( X \) onto \( X' \), we say that \( u \) is an isomorphism.

If \( u : X \to X' \) is a continuous map such that \( p' \circ u = p \), then \( u \) is automatically a local isomorphism.

**Definition 3.3.** Let \( p_0 : \Omega \to \mathbb{C}^n \) be a connected domain and \( S \subseteq H(\Omega) \). Let \( p : X \to \mathbb{C}^n \) be a connected domain and \( \varphi : \Omega \to X \) a continuous map with \( p \circ \varphi = p_0 \). We say \((X, p, \varphi)\) is an \( S \)-extension of \( p_0 : \Omega \to \mathbb{C}^n \) if, to every \( f \in S \), there exists \( F_f \in H(X) \) such that \( F_f \circ \varphi = f \).

Note that \( F_f \) is uniquely determined (first on \( \Omega \) since \( F_f \circ \varphi = f \), hence on \( X \) by the principle of analytic continuation). \( F_f \) is called the extension (or continuation) of \( f \) to \( X \).

**Definition 3.4.** Let \( p_0 : \Omega \to \mathbb{C}^n \) be a connected domain and \( S \subseteq H(\Omega) \). An \( S \)-envelope of holomorphy is an \( S \)-extension \((X, p, \varphi)\) such that the following holds:

\( (*) \) For any \( S \)-extension \((X', p', \varphi')\) of \( p_0 : \Omega \to \mathbb{C}^n \), there is a holomorphic map \( u : X' \to X \) such that \( p \circ u = p' \), \( u \circ \varphi' = \varphi \) and \( F'_f = F_f \circ u \) for all \( f \in S \), where \( F_f, F'_f \) are the extensions of \( f \in S \) to \( X, X' \) respectively.

Note that \( u \) in \( (*) \) is unique (since it is determined on \( \varphi' (\Omega) \) by the equation \( u \circ \varphi' = \varphi \)).

**Remark 3.5.** The \( S \)-envelope of holomorphy, if it exists, is unique up to “isomorphism.” In fact, let \((X, p, \varphi), (X', p', \varphi')\) be two \( S \)-envelopes of holomorphy. Then, by \( (*) \) of Definition 3.4 there are holomorphic maps \( u : X' \to X, v : X \to X' \) such that \( p = p' \circ v, p' = p \circ u, \varphi = u \circ \varphi', \varphi' = v \circ \varphi \). Then \( u \circ v \circ \varphi = u \circ \varphi' = \varphi \), so that \( u \circ v \) is the identity on \( \varphi(\Omega) \) which is open in \( X \). Hence, by the principle of analytic continuation \( u \circ v = \text{identity on } X \). Similarly, \( v \circ u = \text{identity on } X' \). Thus, \( u \) is an isomorphism of \( X' \) onto \( X \) with \( p' = p \circ u, \varphi = u \circ \varphi' \).

**Theorem 3.6** (Thullen). The \( S \)-envelope of holomorphy of any \( S \subseteq H(\Omega) \) exists.

**Proof.** For any \( p_0 : \Omega \to \mathbb{C}^n \) and \( S \subseteq H(\Omega) \), we define a map \( \varphi = \varphi(p_0, S) \) of \( \Omega \) into \( O(S) \) as follows. Let \( a \in \Omega \) and \( a_0 = p_0(a) \in \mathbb{C}^n \). Let \( U \) be an open neighbourhood of \( a \) such that \( p_0|_U \) is an isomorphism onto an open set \( U_0 \subseteq \mathbb{C}^n \). Let \( \{U_0, f_s : s \in S\}_{a_0} \) be the \( S \)-germ at \( a_0 \) defined by the pair \((U_0, \{f_s\}_{s \in S})\), where \( f_s := s \circ (p_0|_U)^{-1} \), \( s \in S \). We set \( \varphi(a) := \{U_0, f_s : s \in S\}_{a_0} \).

Easy to verify that \( \varphi \) is a continuous and that \( p \circ \varphi = p_0 \), where \( p : O(S) \to \mathbb{C}^n \) is the natural projection. In particular, \( \varphi \) is a local isomorphism.

Since \( \Omega \) is connected, so is \( \varphi(\Omega) \). Let \( X \) be the connected component of \( O(S) \) containing \( \varphi(\Omega) \), and denote again by \( p \) the restriction to \( X \) of the map \( p : O(S) \to \mathbb{C}^n \).

We claim that \((X, p, \varphi)\) is an \( S \)-envelope of holomorphy of \( \Omega \).
First, we observe that, for all \( s \in S \), we have a holomorphic function \( F_s \) on \( O(S) \) defined as follows. We set \( F_s(U_0, f_s : s \in S_{1\alpha}) := f_s(a_0) \). Easy to verify that \( F_s \) is holomorphic on \( O(S) \). We denote the restriction of \( F_s \) to \( X \) again by \( F_s \). Now, by the very definition of \( \varphi \), it follows that \( F_s \circ \varphi = s \) for all \( s \in S \).

Let \( p' : X' \to \mathbb{C}^n, \varphi' : \Omega \to X' \) be given with \( p' \circ \varphi' = p_0 \), and suppose that for all \( s \in S \), there exists \( F_s' \in H(X') \) so that \( s = F_s' \circ \varphi' \). Let \( S' = \{ F_s' : s \in S \} \). Let \( u : X' \to O(S) \) be the map \( \varphi(p', S') \) (defined at the beginning of this proof). Since \( F_s' \circ \varphi' = s \) and \( p' \circ \varphi' = p_0 \), we have \( \varphi = u \circ \varphi' \) (locally, \( F_s' \circ p^{-1} = F_s' \circ \varphi' \circ \varphi'^{-1} \circ p'^{-1} = s \circ p_0^{-1} \)). Clearly, \( p' = p \circ u \).

**Definition 3.7.** Let \( p_0 : \Omega \to \mathbb{C}^n \) be a connected domain over \( \mathbb{C}^n \). If \( S = H(\Omega) \), the \( S \)-envelope of holomorphy of \( \Omega \) is called the envelope of holomorphy of \( \Omega \).

**Proposition 3.8.** Let \( p_0 : \Omega \to \mathbb{C}^n \) be a connected domain over \( \mathbb{C}^n \) and \( f \in H(\Omega) \). Let \( F \) be its extension to the envelope of holomorphy \((X, p, \varphi)\). Then \( f(\Omega) = F(X) \). In particular, if \( f \) is bounded, \( |f(x)| \leq M \) for all \( x \in \Omega \), then \( F \) is bounded and \( |F(x)| \leq M \) for all \( x \in X \).

**Proof.** Since \( f = F \circ \varphi \), we have \( f(\Omega) \subseteq F(X) \). Suppose there is a \( c \in F(X) \setminus f(\Omega) \). Then \( \frac{1}{f - c} \in H(\Omega) \). Let \( G \) be its extension to \( X \). Then \( G(F - c) \) is the extension of \( 1 = (f - c)^{-1}(f - c) \) to \( X \), so that \( G(F - c) \equiv 1 \) on \( X \). This implies that \( F(x) \neq c \) for all \( x \in X \), a contradiction.

At this stage, one might like to know how two envelopes of holomorphy are related to each other if the underlying domains are related by a holomorphic mapping. To answer such questions, we require two lemmas. Their proofs are routine and so we shall skip their proofs.

**Lemma 3.9.** Let \( p_0 : \Omega \to \mathbb{C}^n \) be a domain and \( \Psi : \Omega \to \mathbb{C}^n \) be a holomorphic map such that \( \det(d\Psi)(a) \neq 0 \) for some \( a \in \Omega \). Then, there exists a neighbourhood \( U \ni a \) such that \( \Psi|_U \) is an analytic isomorphism.

**Lemma 3.10.** Let \( p_0 : \Omega \to \mathbb{C}^n, p'_0 : \Omega' \to \mathbb{C}^n \) be connected domains over \( \mathbb{C}^n \) and \((X', p', \varphi')\) be the \( T \)-envelope of holomorphy of \( p'_0 : \Omega' \to \mathbb{C}^n, T \subseteq H(\Omega') \). Let \( u : \Omega \to \Omega' \) be a local isomorphism, and let \( S = \{ f \circ u : f \in T \} \). Then \((X', p', \varphi' \circ u)\) is the \( S \)-envelope of \( q_0 : \Omega \to \mathbb{C}^n \) where \( q_0 = p'_0 \circ u \).

**Proposition 3.11.** Let \( p_0 : \Omega \to \mathbb{C}^n, p'_0 : \Omega' \to \mathbb{C}^n \) be connected domains over \( \mathbb{C}^n \), and \((X, p, \varphi),(X', p', \varphi')\) their envelopes of holomorphy. Let \( u : \Omega \to \Omega' \) be a local isomorphism. Then there exists a holomorphic map \( \tilde{u} : X \to X' \) such that \( \varphi' \circ u = \tilde{u} \circ \varphi \).

**Proof.** Let \( v := \varphi' \circ u : \Omega \to X' \). Then \( v \) is holomorphic and a local isomorphism. We have to show that there is \( u : X \to X' \) so that \( u \circ \varphi = v \). Consider the map \( \psi := p' \circ v : \Omega \to \mathbb{C}^n \). Then \( \psi \) is again a local isomorphism. If \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \), the \( \psi_j \)’s are holomorphic. Let \( \eta : \Omega \to \mathbb{C} \) be given by \( \eta(x) := \det(d\psi(x)) \). Then, since \( \psi \) is a local isomorphism, \( \eta(x) \neq 0 \) for all \( x \in \Omega \). Let \( \Psi_j \) be the extension of \( \psi_j \) to \( X \), and let \( \Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n) \). Let \( H \) be the extension of \( \eta \) to \( X \). Then, by the principle of analytic continuation, \( H = \det(d\Psi) \). Moreover, by Proposition 3.8 we have \( H(x) \neq 0 \) for all \( x \in X \). Hence by Lemma 3.9, \( \Psi : X \to \mathbb{C}^n \) is a local isomorphism. Moreover, \( \Psi \circ \varphi = \psi \).
Consider now the domains $\psi : \Omega \rightarrow \mathbb{C}^n$, $p' : X' \rightarrow \mathbb{C}^n$, and $v : \Omega \rightarrow X'$. Let $S = \{f \circ u : f \in H(\Omega')\} = \{F \circ v : F \in H(X')\}$. By Lemma 3.10 $(X', p', v)$ is the $S$-envelope of holomorphy of $\psi : \Omega \rightarrow \mathbb{C}^n$. Now any holomorphic function on $\Omega$ can be extended to $X$, so that $(X, p, \Psi)$ is an $S$-extension of $\psi : \Omega \rightarrow \mathbb{C}^n$. Therefore there exists a holomorphic map $\tilde{u} : X \rightarrow X'$ such that $p' \circ \tilde{u} = \Psi$ and $\tilde{u} \circ \varphi = v$.

**Definition 3.12.** Let $p_0 : \Omega \rightarrow \mathbb{C}^n$ be a connected domain over $\mathbb{C}^n$ and $S \subseteq H(\Omega)$. $\Omega$ is called an $S$-domain of holomorphy if the natural map of $\Omega$

$$\varphi : x \mapsto [p_0(U_x), f \circ (p_0|U_x)^{-1}]_{p_0(x)}$$

into its $S$-envelope of holomorphy is an analytic isomorphism. If $S = H(\Omega)$, $\Omega$ is called a domain of holomorphy.

The reader would intuit that the envelope of holomorphy of a connected domain $p_0 : \Omega \rightarrow \mathbb{C}^n$ is a domain of holomorphy. We now give the proof behind this intuition.

**Corollary 3.13.** If $(X, p, \varphi)$ is the envelope of holomorphy of $p_0 : \Omega \rightarrow \mathbb{C}^n$ then $p : X \rightarrow \mathbb{C}^n$ is a domain of holomorphy.

**Proof.** In Lemma 3.10 if we replace $\Omega'$ by $X$, $T$ by $\{F_f : f \in H(\Omega)\}$ and $u$ by $\varphi$ then we get $S = H(\Omega)$. Therefore by Lemma 3.10 we get the desired result. □

Consider a domain $\Omega \subseteq \mathbb{C}^n$ (in which case $p_0$ is just the inclusion map). If such an $\Omega$ is a domain of holomorphy, then combining the above corollary with the details of Thullen’s construction in Theorem 3.6, it is easy to deduce the following equivalent definition for $\Omega \subseteq \mathbb{C}^n$ to be a domain of holomorphy.

**Definition 3.14.** Let $\Omega \subseteq \mathbb{C}^n$. We say $\Omega$ is a domain of holomorphy if there do not exist nonempty open sets $U_1, U_2$, with $U_2$ connected, $U_2 \not\subseteq \Omega$, $U_1 \subset U_2 \cap \Omega$ such that to every $f \in H(\Omega)$, there exists $F_f \in H(U_2)$ satisfying $F_f|_{U_1} = f|_{U_1}$.

### 3.2 Domains of holomorphy

Now that we have introduced the concept of a domain of holomorphy, it would be interesting to see if one can characterize when a connected domain $p_0 : \Omega \rightarrow \mathbb{C}^n$ is a domain of holomorphy. This is quite technical, even when $\Omega \subseteq \mathbb{C}^n$. In the latter case, there are several equivalent characterizations. A characterization that is easy to check would require more space than is available in this report. But we shall aim to provide one characterization.

**Definition 3.15.** Let $p_0 : \Omega \rightarrow \mathbb{C}^n$ be a connected domain over $\mathbb{C}^n$, $f \in H(\Omega)$ and $A \subseteq \Omega$. We write

$$\|f\|_A := sup_{x \in A} |f(x)|.$$

- If $A \subseteq \Omega$, and $S \subseteq H(\Omega)$, we set

$$\hat{A}_S := \{x \in \Omega : |f(x)| \leq \|f\|_A \text{ for all } f \in S\}.$$
• If \( S = H(\Omega) \), we simply write \( \hat{A} := \hat{A}_S \).

We shall need the following technical, but elementary, lemma.

Lemma 3.16. Let \( A \subseteq \Omega \) such that \( \|f\|_A < \infty \) for any \( f \in H(\Omega) \). Then, there is a compact set \( K \subseteq \Omega \) such that \( A \subseteq \hat{K} \).

Lemma 3.17. The following two statements are equivalent.

(a) For any \( K \subseteq \Omega \), \( K \) compact, \( \hat{K} \) is also compact.

(b) For any (infinite) sequence \( (x_n) \subseteq \Omega \) which has no limit point in \( \Omega \) there exists \( f \in H(\Omega) \) such that \( \{f(x_n)\} \) is unbounded.

Proof. (a) \( \Rightarrow \) (b). Let \( (x_n) \) be a sequence without limit point in \( \Omega \). Then \( \{x_n\} \notin \hat{K} \) for any compact set \( K \). By Lemma 3.16 there exists \( f \in H(\Omega) \) such that \( \{f(x_n)\} \) is not bounded.

(b) \( \Rightarrow \) (a). If \( \hat{K} \) is not compact, there exists a sequence \( \{x_n\} \subseteq \hat{K} \), which has no limit point in \( \hat{K} \) and therefore in \( \Omega \) as \( \hat{K} \) is closed in \( \Omega \). Let \( \hat{f} \in H(\Omega) \) such that \( \{f(x_n)\} \) is unbounded. Then \( \|f\|_{\hat{K}} = \infty \). But, it follows from the definition of \( \hat{K} \) that \( \|f\|_{\hat{K}} = \|f\|_K < \infty \). \( \square \)

If the conditions of Lemma 3.17 are satisfied we say that \( \Omega \) is \textit{holomorphically convex}.

We now extend the notion of a polydisc to arbitrary domains over (i.e. not necessarily contained in) \( \mathbb{C}^n \). Such a notion would enable us to define, for any point \( a \in \Omega \), a notion of “distance from the boundary” even when \( \Omega \) is not a domain contained in \( \mathbb{C}^n \). Readers will note that the extended notion of a polydisc will result in the polydiscs \( D^n(a, r) \) when \( \Omega \subseteq \mathbb{C}^n \), and that the notion of “distance from the boundary” will coincide with the distance from \( \partial \Omega \) with respect to the \( |\cdot| \)-norm when \( \Omega \subseteq \mathbb{C}^n \).

To distinguish these new objects from the classical polydiscs, we shall use the notation \( P(a, r) \). To be precise, we make a definition.

Definition 3.18. Let \( p_0 : \Omega \to \mathbb{C}^n \) be a connected domain over \( \mathbb{C}^n \), \( a \in \Omega \). A \textit{polydisc of radius} \( r \) about \( a \) is a connected open set \( U \) containing \( a \) such that \( p_0|_U \) is an analytic isomorphism onto the set \( \{z \in \mathbb{C}^n : |z_j - b_j| < r \} \), where \( p_0(a) = (b_1, b_2, \ldots, b_n) \). We denote the set \( U \) by \( P(a, r) \). The maximal polydisc around \( P(a, r_0) \) is the union of all polydiscs about \( a \).

Lemma 3.19. \( P(a, r_0) \) is a polydisc about \( a \) of radius

\[
r_0 = \sup r
\]

where the supremum is over all polydiscs \( P(a, r) \) about \( a \).

Proof. It suffices to show that the map

\[
p_0 : P(a, r_0) \to D^n(b, r_0)
\]

is bijective. By definition \( p_0(P(a, r_0)) \subset D^n(b, r_0) \).

\( p_0 \) injective: If \( x, x' \in P(a, r_0) \), there is a polydisc \( P(a, r) \) containing both \( x, x' \), so that \( p_0(x) \neq p_0(x') \).

\( p_0 \) surjective: If \( z \in D^n(b, r_0) \), then \( \max |z_j - b_j| < r_0 \), hence there is a polydisc \( P(a, r) \), such that \( |z_j - b_j| < r \leq r_0 \), so that there is a point \( x \in P(a, r) \) with \( p_0(x) = z \). \( \square \)
Definition 3.20. The radius of the maximal polydisc about \( a \) is called the distance of \( a \) from the boundary of \( \Omega \) and is denoted by \( d(a) \).

Lemma 3.21. If there is a point \( a \in \Omega \) with \( d(a) = \infty \), then \( p_0 \) is an isomorphism of \( \Omega \) onto \( \mathbb{C}^n \).

Proof. Let \( S = \{ x \in \Omega : d(x) = \infty \} \). By assumption \( S \neq \emptyset \). By definition \( S \) is open as \( d(a) = \infty \) implies that there is an open set \( U \) containing \( a \) such that \( p_0|_U \) is an isomorphism onto \( \mathbb{C}^n \). Let \( x_n \in S, x_n \to x \). Let \( P(x, r) \) be a polydisc about \( x \). Now we consider \( x_n \) such that \( x_n \in P(x, r) \). Now as \( x_n \in S \) there exists \( U \) containing \( x_n \) such that \( p_0|_U \) is isomorphism onto \( \mathbb{C}^n \). Therefore \( P(x, r) \subset U \), and hence \( x \in U \). Therefore \( x \in S \), and hence \( S \) is closed. As \( \Omega \) is a domain we have \( S = \Omega \). Therefore \( p_0 \) is a covering. As \( \mathbb{C}^n \) is simply connected we have that \( p_0 : \Omega \to \mathbb{C}^n \) is an isomorphism.

Definition 3.22. If \( A \subset \Omega \), we set

\[
\begin{align*}
d(A) := \inf_{a \in A} d(a).
\end{align*}
\]

Remark 3.23. Since \( d \) is continuous (which follows from definition), if \( K \) is a compact subset of \( \Omega \) then \( d(K) > 0 \).

The following is a very important result that we shall make use of in this and the next chapter.

Proposition 3.24. Let \( p_0 : \Omega \to \mathbb{C}^n \) be a domain. Let \( K \) be a compact subset of \( \Omega \) and \( x_0 \in \hat{K} \). Let \( a = p_0(x_0) \), and let \( V \) be a polydisc about \( x_0 \), and \( D = p_0(V) \). Then, for any \( f \in H(\Omega) \), if \( g := f \circ (p_0|_V)^{-1} \), the series

\[
\sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g(a)}{\alpha!} (z - a)^\alpha
\]

converges in the polydisc \( D^\alpha(a,d(K)) \).

Proof. Note that clearly \( g \in H(D) \). Let \( 0 < r < d(K) \). For any \( x \in K \), we consider the compact set, \( K' = \bigcup_{x \in K} \overline{P(x, r)} \). Let \( M = \|f\|_{K'} \). By Cauchy’s inequality applied to \( f|_{P(x, r)} \), we have \( |D^\alpha f(x)| \leq M \alpha! r^{-|\alpha|}, x \in K \), so that \( ||D^\alpha f||_K \leq M \alpha! r^{-|\alpha|} \). Hence, by definition of \( \hat{K} \), we have

\[
||D^\alpha f||_{\hat{K}} \leq M \alpha! r^{-|\alpha|}.
\]

Since \( x_0 \in \hat{K} \), this implies that \( |D^\alpha g(a)| \leq M \alpha! r^{-|\alpha|} \). Therefore the series \( \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g(a)}{\alpha!} (z - a)^\alpha \) converges on \( D^\alpha(a, r) \). Since \( r < d(K) \) is arbitrary, the result follows.

Before we prove the main result of this section, we state a simple

Lemma 3.25. Let \( p_0 : \Omega \to \mathbb{C}^n \) be a domain and let \( S \subseteq H(\Omega) \) has the property that for each \( f \in S \) and \( \alpha \in \mathbb{N}^n \), \( D^\alpha f \in S \). Let \( (X, p, \varphi) \) be the \( S \)-envelope of holomorphy of \( p_0 : \Omega \to \mathbb{C}^n \). Then, \( \varphi \) is injective if and only if \( S \) separates points of \( p_0^{-1}(p_0(a)) \) for every \( a \in \Omega \). In particular, if \( p_0 : \Omega \to \mathbb{C}^n \) is a domain of holomorphy, then \( H(\Omega) \) separates points of \( \Omega \).

The above is an elementary consequence of identifying \( X \) with a special connected component of \( O(S) \) as done in Thullen’s construction.

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Theorem 3.26 (H. Cartan-P. Thullen). Suppose that \( p_0 : \Omega \to \mathbb{C}^n \) is a domain of holomorphy. Then, for any compact set \( K \subset \Omega \), we have

\[
d(K) = d(\hat{K}).
\]

Proof. Clearly \( d(K) \geq d(\hat{K}) \) as \( K \subset K' \). Suppose that strict inequality holds. Then, there is \( x_0 \in \hat{K} \) such that \( d(x_0) = r_0 < \rho = d(K) \). Let \( a = p_0(x_0) \) and let \( D_0 = \{ z \in \mathbb{C}^n : |z - a| < d(K) = \rho \} \). Let \( \Omega_0 \) be the connected component of \( p_0^{-1}(D_0) \) containing \( x_0 \). We assert that \( p_0|_{\hat{\Omega}_0} \) is injective. In fact, it follows from Proposition 3.24 that for any \( f \in H(\Omega) \), there is \( g_f \in H(D_0) \) such that \( f|_{\hat{\Omega}_0} = g_f \circ p_0 \). On the other hand, since \( p_0 : \Omega \to \mathbb{C}^n \) is a domain of holomorphy, by Lemma 3.25, since \( g_f \circ p_0 \) takes the same value at any two points \( x, y \in \Omega_0 \) with \( p_0(x) = p_0(y) \), it follows that \( p_0|_{\hat{\Omega}_0} \) is injective.

Let \( r_0 < r < \rho \) and \( \Omega_1 = \{ x \in \Omega_0 : |p_0(x) - a| < r \} \), and let \( Y \) be the disjoint union of \( \Omega \) and \( D \), where \( D = \{ z \in \mathbb{C}^n : |z - a| < r \} \). We define an equivalence relation on \( Y \) by the requirement that \( z \in D \) is equivalent to at most one point \( x \in \Omega \), and that, if and only if \( x \in \Omega_1 \) and \( p_0(x) = z \). Let \( X \) be the quotient of \( Y \) by this equivalence relation. Then \( X \) is Hausdorff, and the map of \( Y \) into \( \mathbb{C}^n \) which is \( p_0 \) on \( \Omega \) and the inclusion of \( D \) into \( \mathbb{C}^n \) induces a local homeomorphism \( p : X \to \mathbb{C}^n \).

Moreover, the inclusion of \( \Omega \) in \( Y \) induces a map \( \varphi : \Omega \to X \) such that \( p \circ \varphi = p_0 \). We claim that for any \( f \in H(\Omega) \), there is \( F_f \in H(X) \) such \( F_f \circ \varphi = f \). In fact, since there is \( g_f \in H(D_0) \) such that \( f_{\Omega_0} = g_f \circ p_0 \), by Proposition 3.24, we define a function \( G_f \) on \( X \) by \( G_f|_{\Omega} = f, G_f|_{D} = g_f|_{D} \).

This induces \( F_f \in H(X) \) and we clearly have \( F_f \circ \varphi = f \). Hence \( (x, p, \varphi) \) is an \( H(\Omega) \)-extension of \( p_0 : \Omega \to \mathbb{C}^n \). Since \( p_0 : \Omega \to \mathbb{C}^n \) is, by assumption, a domain of holomorphy, it follows that \( \varphi \) is an analytic isomorphism. In particular, since \( X \) contains a polydisc of radius \( r \) about \( x_0 \), \( \Omega \) contains a polydisc of radius \( r \) about \( x_0 \), contradicting our assumption that \( r > r_0 = d(x_0) \). \( \square \)

The same reasoning can be used to prove the following:

Theorem 3.27 (Cartan-Thullen). Let \( p_0 : \Omega \to \mathbb{C}^n \) be a domain. Let \( S \subseteq H(\Omega) \) be a subalgebra of \( H(\Omega) \) containing the functions \( p_1, ..., p_n \) \((p_0 = (p_1, ..., p_n)) \) and closed under differentiation \((i.e., f \in S \text{ implies } D^\alpha f \in S \text{ for all } \alpha \in \mathbb{N}^n) \). Then, if the natural map of \( \Omega \) into its \( S \)-envelope of holomorphy is an isomorphism, we have \( d(K) = d(\hat{K}_S) \) for any compact \( K \subset \Omega \).

Corollary 3.28. If \( \Omega \) is an open set in \( \mathbb{C}^n \) which is a domain of holomorphy and \( p_0 \) is the inclusion of \( \Omega \) in \( \mathbb{C}^n \), then for any compact set \( K \subset \Omega \), \( \hat{K} \) is also compact.

Proof. \( \hat{K} \) is clearly closed in \( \Omega \). Moreover, since \( d(K) = d(\hat{K}) \), it follows that \( \hat{K} \) is closed in \( \mathbb{C}^n \) (since the closure of \( \hat{K} \) in \( \mathbb{C}^n \) cannot meet \( \partial \Omega \)). Moreover, \( \hat{K} \) is contained in the polydisc \( \{ z \in \mathbb{C}^n : |z| \leq \rho \} \) where \( \rho = \max_j |z_j|_K \), and so is bounded. Hence \( \hat{K} \) is compact. \( \square \)

We are now quite close to characterizing domains of holomorphy when the domains lie in \( \mathbb{C}^n \).

Proposition 3.29. Suppose \( \Omega \not\subseteq \mathbb{C}^n \) is a connected open subset and has the property that for any compact set \( K \subset \Omega \), \( \hat{K} \) is also compact. Then \( \Omega \) is a domain of holomorphy.
Proof. Since $\Omega \subset \mathbb{C}^n$ it is possible to construct a countable subset $Z = \{x_\nu : \nu \in \mathbb{Z}_+\}$ of $\Omega$ that contains no limit points in $\Omega$ and such that $\partial \Omega \subset \bar{Z}$. Now by Lemma 3.17 there exists $F \in H(\Omega)$ such that $F$ is unbounded on $Z$. Now, for each $z \in \partial \Omega$, there is a subsequence $\{x_{\nu_k}\}_{k \in \mathbb{N}}$ such that $x_{\nu_k} \to z$. Thus, as $z \in \partial \Omega$ was arbitrary, we conclude that for any domain $U$ such that $U \cap \partial \Omega \neq \emptyset$, $F$ is unbounded on $U \cap \Omega$. Now if $U_2$ is a domain in $\mathbb{C}^n$ such that $U_2 \cap \Omega \neq \emptyset$ and $U_2 \not\subset \Omega$ then $U_2 \cap \partial \Omega \neq \emptyset$. Therefore by the last two assertions and by Definition 3.14, $\Omega$ is a domain of holomorphy. □

Summarizing from Corollary 3.28 and the above proposition:

**Theorem 3.30.** Let $\Omega \subset \mathbb{C}^n$ be a connected open subset. Then, $\Omega$ is a domain of holomorphy if and only if it is holomorphically convex.

5This is because of $(ii)$ above. This allows us to use Proposition
Chapter 4

More characterizations of domains of holomorphy in $\mathbb{C}^n$

Chapter 3 ended with a characterization of a proper subdomain of $\mathbb{C}^n$ to be a domain of holomorphy. Unfortunately, when $n \geq 2$, the condition of holomorphic convexity is very hard to check. It would thus be desirable to have other (equivalent) characterizations that are more amenable to computation. It is with this aim that we present a long theorem in Section 4.1 that presents several equivalent characterizations for a domain $\Omega \subset \mathbb{C}^n$ to be a domain of holomorphy.

The proof of this theorem, i.e. Theorem 4.8, is not easy. The proof of one of the implications requires the technical and very sophisticated result of Hörmander that the $\overline{\partial}$-problem has a solution, with not necessarily compactly-supported data, on a pseudoconvex domain. This will be clarified in greater detail just after the statement of Theorem 4.8. Hörmander’s work is a bit beyond the scope of this survey. Therefore, we shall present proofs of only some of the parts of Theorem 4.8.

4.1 Further characterizations of domains of holomorphy

We begin this section with an application of Theorem 3.30.

**Theorem 4.1.** Let $\Omega$ be a convex domain in $\mathbb{C}^n$. Then, $\Omega$ is a domain of holomorphy.

**Proof.** Without loss of generality, assume $0 \in \Omega$. Define:

$$K_\nu := (1 - \frac{1}{\nu}) \overline{\Omega \cap \mathbb{B}^n(0, \nu)}$$

for each $\nu (\geq 2) \in \mathbb{Z}_+$. Then $K_\nu$’s are convex sets. Now pick a compact subset $L \subset \Omega$. Then there exists $\nu(L) \in \mathbb{Z}_+$ such that $L \subset K_\nu$ for all $\nu \geq \nu(L)$. Let $z_0 \notin K_{\nu(L)}$. Then, there exists a real linear functional on $\mathbb{R}^{2n}$, $\Lambda$, such that

$$\Lambda(z_0) = 1, \quad \sup_{z \in K_{\nu(L)}} \Lambda(z) = \alpha < 1.$$  \hspace{1cm} (4.1)
Now write \( z_j = x_j + iy_j \) for \( j = 1, \ldots, n \). We can write \( \Lambda(z) = \sum_{j=1}^n a_jx_j + b_jy_j \) for some \( a_j, b_j \in \mathbb{R} \). Now let

\[
\lambda(z) := \sum_{j=1}^n (a_j - ib_j)z_j.
\]

Note that \( \text{Re} \, \lambda(z) = \Lambda(z) \). Now define \( \lambda(z) : \Omega \rightarrow \mathbb{C} \) for some \( a_j, b_j \in \mathbb{R} \).

Now let

\[
\lambda(z) := n \sum_{j=1}^n (a_j - ib_j)z_j.
\]

Note that \( \text{Re} \, \lambda(z) = \Lambda(z) \). Now define \( F(z) := e^{\lambda(z)} \), \( z \in \mathbb{C}^n \). By (4.1) we have

\[
|F(z_0)| = e > e^\alpha = \sup_{K_{\nu(L)}} |F| \geq \sup_L |F|.
\]

Hence if \( z_0 \notin K_{\nu(L)} \), then \( z_0 \notin \tilde{L}_\Omega \). Therefore \( \Omega \ni K_{\nu(L)} \supset \tilde{L}_\Omega \). This implies \( \tilde{L}_\Omega \) is compact. Hence, by Theorem 3.30, \( \Omega \) is a domain of holomorphy as \( L \) is an arbitrary compact subset of \( \Omega \).

**Definition 4.2.** A function \( \mu : \mathbb{C}^n \rightarrow \mathbb{R} \) is called a distance functional if:

1. \( \mu \geq 0 \) and \( \mu(z) = 0 \) if and only if \( z = 0 \);
2. \( \mu(tz) = |t| \mu(z) \) for all \( t \in \mathbb{C} \) and for all \( z \in \mathbb{C}^n \); and
3. \( \mu \) is continuous.

Given a distance functional \( \mu \), we can talk about the “distance from \( \partial \Omega \)” of a point \( z \in \Omega \) measured in terms of \( \mu \). This motivates the next definition.

**Definition 4.3.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \), and let \( \mu : \mathbb{C}^n \rightarrow \mathbb{R} \) be a distance functional. For each \( z \in \Omega \) we define

\[
\mu_\Omega(z) := \inf\{\mu(z - w) : w \in \Omega^C\}. \tag{4.2}
\]

If \( X \) is a subset of \( \Omega \), then we write

\[
\mu_\Omega(X) := \inf\{\mu_\Omega(x) : x \in X\}.
\]

In the above definition, we do not distinguish between proper subdomains and the case when \( \Omega = \mathbb{C}^n \). From the right-hand side of (4.2), we see that for some \( z \in \Omega \), \( \mu_\Omega(z) = \infty \iff \Omega = \mathbb{C}^n \). This is reminiscent of Lemma 3.21. In fact, if the \( \Omega \) of Lemma 3.21 were a domain in \( \mathbb{C}^n \) (with \( p_0 \) just being the inclusion map), then the entire discussion in Chapter 3 could be viewed as one where the \( L^\infty \)-norm on \( \mathbb{C}^n \) is the distance functional of choice. In a sense, parts of the theorem below arise from the flexibility we get by considering various distance functionals and not just the \( L^\infty \)-norm.

Perhaps the most important concept in studying domains of holomorphy is that of plurisubharmonic functions.

We recall that a \( C^2 \)-smooth function \( h \) in an open set \( \Omega \subset \mathbb{C} \) is called harmonic if \( \Delta h = 4\partial^2 h/ \partial z \partial \overline{z} = 0 \) in \( \Omega \).

**Definition 4.4.** A function \( u \) defined in an open set \( \Omega \subset \mathbb{C} \) and with values in \( [-\infty, +\infty) \) is called subharmonic if

1. \( u \) is upper semicontinuous, that is, \( \{z : z \in \Omega, u(z) < s\} \) is open for every real number \( s \);
ii) For every compact set $K \subset \Omega$ and every continuous function $h$ on $K$ which is harmonic in the interior of $K$ and is $\geq u$ on the boundary of $K$, we have $u \leq h$ in $K$.

**Definition 4.5.** A function $u$ defined in an open set $\Omega \subset \mathbb{C}^n$ and with values in $[−\infty, +\infty)$ is called *plurisubharmonic* if

1. $u$ is upper semicontinuous;
2. For every arbitrary $z$ and $w \in \mathbb{C}^n$, the function $\tau \mapsto u(z + \tau w)$ is subharmonic in its domain of definition.

We need a couple of final definitions before we can state Theorem 4.8. The definitions give two notions of “convexity” for domains in $\mathbb{C}^n$. One of the outcomes of Theorem 4.8 is that the two notions coincide for domains with $C^2$-smooth boundaries.

**Definition 4.6.** Let $\Omega$ be a domain in $\mathbb{C}^n$. We say that $\Omega$ is *Hartogs pseudoconvex* if there exists a distance functional $\mu: \mathbb{C}^n \to [0, \infty)$ such that the function $−\log \mu_\Omega$ is plurisubharmonic on $\Omega$.

**Definition 4.7.** Let $\Omega \subset \mathbb{C}^n$ be a domain having $C^2$-smooth boundary and let $\rho$ be a defining function for $\Omega$, i.e. $\rho: V \to \mathbb{R}$ is a $C^2$-smooth function, where $V$ is an open neighbourhood of $\overline{\Omega}$, such that

- $\Omega = \{z \in V : \rho(z) < 0\}$; and
- $\nabla \rho(z) \neq 0 \forall z \in \partial \Omega$.

We say that $\Omega$ is *Levi pseudoconvex* if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z)v_j\bar{v}_k \geq 0 \quad \text{for all } V = (v_1, \ldots, v_n) \in T_z(\partial \Omega) \cap iT_z(\partial \Omega), \text{ and for all } z \in \partial \Omega. \quad (4.3)$$

Note that, in (4.3), we are viewing the tangent spaces to $\partial \Omega$ *extrinsically* as real subspaces in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Thus, $T_z(\partial \Omega) \cap iT_z(\partial \Omega)$ denotes the largest $\mathbb{R}$-linear subspace of $T_z(\partial \Omega)$ that is $\mathbb{C}$-linear in the ambient $\mathbb{C}^n$. We shall abbreviate $T_z(\partial \Omega) \cap iT_z(\partial \Omega) := H_z(\partial \Omega)$.

We now have all the definitions needed to state the main theorem of this chapter.

**Theorem 4.8.** Let $\Omega$ be a domain in $\mathbb{C}^n$. Then, the following statements are equivalent.

1) $\Omega$ is a domain of holomorphy;

2) For any distance functional $\mu: \mathbb{C}^n \to [0, \infty)$, the function $−\log \mu_\Omega$ is plurisubharmonic on $\Omega$;

3) $\Omega$ is Hartogs pseudoconvex;

4) $\Omega$ admits a continuous exhaustion function $u$, i.e. given any $c \in \mathbb{R}, \{z \in \Omega : u(z) < c\}$ is a relatively compact subset of $\Omega$;
5) \( \Omega \) is holomorphically convex.

Furthermore if \( \Omega \) has \( C^2 \)-smooth boundary, then the following is equivalent to the five statements above:

(6) \( \Omega \) is Levi pseudoconvex.

A sketch of the proof. The scheme of the proof of the equivalence of the first five statements above is summarized by the following diagram of implications:

![Diagram](image)

The proofs of some of the above implications are very technical. Of these, the proof that
(3) \( \Rightarrow \) (5) actually relies on the very sophisticated result of Hörmander that if \( \Omega \) is Hartogs pseudoconvex, then for every smooth \((0,q)\)-form \( F \) on \( \Omega \) \((F\) not necessarily compactly supported), \( q = 1, \ldots, n - 1 \), such that \( \bar{\partial}F = 0 \), there exists a \( u \in C^\infty_{c(0,q-1)}(\Omega) \) such that \( \bar{\partial}u = F \). The details are given in [7, Theorem 4.2.8-9]; to the best of my knowledge, there is no other approach to proving (3) \( \Rightarrow \) (5). As stated above, the solvability of the \( \bar{\partial} \)-problem in this generality is a deep result, which is beyond the scope of this survey. Hence, we shall present proofs of only a few selected implications.

We shall discuss some aspects of the equivalence of Levi pseudoconvexity, in case \( \Omega \) has \( C^2 \)-smooth boundary, with the statements (1)–(5) in greater detail in Section 4.2. There are several different proofs of this last equivalence. We will present a proof that is in keeping with the theme — i.e. analytic continuation — of this survey.

(2) \( \Rightarrow \) (3): Given (2), (3) follows from the definition of Hartogs pseudoconvexity.

(3) \( \Rightarrow \) (4): By (3) we know that there exists a distance functional \( \mu \) such that \(-log\mu_\Omega(z)\) is plurisubharmonic on \( \Omega \). Now if \( \Omega \) is unbounded then it is possible that \(-log\mu_\Omega(z)\) is not a exhaustion function (although, by hypothesis, it is plurisubharmonic). Note that if \( \{z_\nu\}_{\nu \in \mathbb{Z}_+} \) is a sequence that approaches \( \partial \Omega \), then \(-log\mu_\Omega(z_\nu) \to \infty \) as \( \nu \to \infty \), but, if \( \Omega \) is unbounded, then the sets \( \{z \in \Omega : -log\mu_\Omega(z) < c\} \) could be unbounded. Therefore if we set \( u(z) := \|z\|^2 - log \mu_\Omega(z) \) where \( \mu \) is given by (3) we get the desired implication (it is easy to check that \( \|z\|^2 \) is plurisubharmonic).

(5) \( \Rightarrow \) (1): We already have proved this in Theorem 3.30. □
4.2 Domains of holomorphy that have smooth boundary

Before we embark upon the proof of the last stage of Theorem 4.8, we need to answer a question about whether Levi pseudoconvexity is, as per Definition 4.7, well defined. In other words, we must ascertain whether (4.3) holds true independently of the choice of the defining function. We establish this in the following proposition.

**Proposition 4.9.** Let \( \Omega \subset \mathbb{C}^n \) be an open set with \( C^2 \)-smooth boundary having a defining function \( \rho \). Let \( \rho_1 \) be another defining function; let \( \Omega = \{ z : \rho_1(z) < 0 \} \) where \( \rho_1 \) is in \( C^2 \) in a neighbourhood of \( \overline{\Omega} \) and \( \nabla \rho_1 \neq 0 \) on \( \partial \Omega \). If \( \Omega \) is Levi pseudoconvex, then

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \rho_1}{\partial z_j \partial \overline{z}_k}(z)v_j v_k \geq 0 \quad \text{for all } V = (v_1, \ldots, v_n) \in H_z(\partial \Omega), \text{ and for all } z \in \partial \Omega. \tag{4.4}
\]

**Proof.** Let us pick a point \( p \in \partial \Omega \). Then, there is a small ball \( B_p \) around \( p \) such that \( (\rho_1/\rho)(z) > 0 \) for all \( z \in B_p \). Now, using a partition-of-unity argument, we may assume that if \( \rho_1 \) is as stated in Proposition 4.9 then \( \rho_1 = h \rho \) with \( h > 0 \) in a neighbourhood of \( \overline{\Omega} \). Hence for all \( z \in \partial \Omega \),

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \rho_1}{\partial z_j \partial \overline{z}_k}(z)v_j v_k = h(z) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z)v_j v_k, \quad \text{provided } V = (v_1, \ldots, v_n) \in H_z(\partial \Omega).
\]

This is because \( \rho|_{\partial \Omega} \equiv 0 \) and \( V = (v_1, \ldots, v_n) \in H_z(\partial \Omega) \iff \sum_{j=1}^{n} \partial_{\overline{z}_j} \rho(z)v_j = 0 \). Hence the proposition. \( \square \)

We will now look at one of the two implications that establish the final stage of Theorem 4.8. Under the hypothesis that \( \partial \Omega \) is \( C^2 \)-smooth, one may achieve this by showing that \( (2) \Rightarrow (6) \Rightarrow (3) \). But this proof has several intermediate steps of a technical nature that make the proof quite lengthy and obscure the main idea behind the machinery. We would like to bring out this clearly. Hence, we shall prove \( (1) \Rightarrow (6) \) directly under the slightly stronger hypothesis that \( \partial \Omega \) is \( C^3 \)-smooth.

**Theorem 4.10.** Let \( \Omega \) be a domain with \( C^3 \)-smooth boundary. If \( \Omega \) is a domain of holomorphy, then \( \Omega \) is Levi pseudoconvex.

**Proof.** Suppose there is a point \( p \in \partial \Omega \) such that there exists a vector \( V \in H_p(\partial \Omega) \) and

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p)v_j v_k < 0,
\]

where \( V = (v_1, \ldots, v_n) \), and \( \rho \) is some choice of defining function for \( \Omega \). This proof depends on making several holomorphic change of coordinates. We shall carefully argue what happens with the first of these changes of coordinates. This will serve as a model for the assertions following subsequent coordinate changes.
Without loss of generality, we may assume that \( \| V \|=1=\| \nabla \rho (p) \| \). Now, let \( \langle \cdot , \cdot \rangle \) denote the standard Hermitian inner-product on \( \mathbb{C}^n \) and let \( T \) be a unitary transformation with respect to \( \langle \cdot , \cdot \rangle \) such that \( T(\nabla \rho (p))=(0,\ldots,i),\ T(V)=(1,0,\ldots,0) \) and \( T(H_{\rho}(\partial \Omega))=\{z_n=0\} \). Now if

\[
(z'_1,z'_2,\ldots,z'_n)=T(z-p),
\]

\[
\rho'(z')=\rho(T^{-1}(z')+p),
\]

then the above represents a holomorphic change of coordinate, say \( \psi \), and \( \rho' \) is the defining function of \( \psi(\Omega) \). By Taylor’s theorem:

\[
\rho'(z')=2\text{Re}\left\{ \sum_{j=1}^{n} \frac{\partial \rho'}{\partial z_j}(0)z'_j \right\} + \text{Re}\left\{ \sum_{j,k=1}^{n} \frac{\partial^2 \rho'}{\partial z'_j \partial z'_k}(0)z'_jz'_k \right\}
+ \sum_{j,k=1}^{n} \frac{\partial^2 \rho'}{\partial z'_j \partial \bar{z}_k}(0)z'_j\bar{z}_k + O(|z'|^3). \tag{4.6}
\]

If we view vectors \( U, W \in \mathbb{C}^n \) as vectors in \( \mathbb{R}^{2n} \), then define

\[
U \cdot W = \sum_{j=1}^{n} \text{Re} u_j \text{Re} w_j + \text{Im} u_j \text{Im} w_j = \text{Re}[\langle U, W \rangle].
\]

From this, and applying the chain rule to (4.5), we get

\[
\rho'(z') = \nabla \rho (p) \cdot T^{-1}(z') + Q(z') + O(|z'|^3),
\]

where \( Q \) is the quadratic form seen in (4.6). If we write \( z'_j = x'_j + iy'_j, \ j=1,\ldots,n \), then, by the properties of \( T \),

\[
\nabla \rho (p) \cdot T^{-1}(z') = \text{Re}[\langle \nabla \rho (p), -iz'_n \nabla \rho (p) \rangle] = y'_n.
\]

This argument shows that, to simplify notations, we may assume without loss of generality that

- \( p = 0; \)
- \( \rho(z) = y_n + Q(z) + O(|z|^3) \), where \( Q \) is the quadratic form given by

\[
Q(z) = \text{Re}\left\{ \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0)z_jz_k \right\} + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(0)z_j\bar{z}_k;
\]
- \( \text{The } V \text{ occurring in our assumption is } (1,0,\ldots,0). \)

Note that these imply that:

\[
\frac{\partial^2 \rho}{\partial z_1 \partial \bar{z}_1}(0) < 0. \tag{4.7}
\]
Let us now make another change of variable

\[(z'_1, \ldots, z'_n) = \Psi(z) = \left( z_1, \ldots, z_{n-1}, i \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(0)z_j z_k \right) . \]

Let \( \Psi = (\Psi_1, \ldots, \Psi_n) \). Then we observe that

\[ \frac{\partial \Psi_i}{\partial z_j}(0) = \delta_{ij} \text{ for } i, j = 1, \ldots, n-1 \text{ and } \frac{\partial \Psi_n}{\partial z_n}(0) = 1. \]

Therefore \( \text{Jac}_C(\Psi)(0) = 1 \). By inverse function theorem for holomorphic functions we have that there exists \( \omega \), a neighbourhood of \( 0 \) in \( \mathbb{C}^n \), such that \( \Psi|_{\omega} \) is a biholomorphism.

After this change of variables the Taylor expansion of \( \rho \) assumes the simpler form

\[ \rho = \text{Im} z'_n + \sum_{j,k=1}^{n} A_{jk} z_j \overline{z}_k + O(||z'||^3). \]

Again, to simplify notations we may therefore, without loss of generality, assume that the original coordinates were choosen so that

\[ \rho = \text{Im} z_n + \sum_{j,k=1}^{n} A_{jk} z_j \overline{z}_k + O(||z||^3), \]  \hspace{1cm} (4.8)

where \( A_{jk} \) is a Hermitian symmetric matrix. Note that, By (4.7), \( A_{11} < 0 \). By (4.8) we have

\[ \rho(z_1, 0, \ldots, 0) = A_{11}|z_1|^2 + O(|z_1|^3). \]  \hspace{1cm} (4.9)

Now, since \( A_{11} < 0 \), we can first choose \( \delta > 0 \), then \( \epsilon > 0 \), and finally \( r << \delta \) such that

- \( D(0, \delta) \times D^{n-1}((0, \ldots, 0, -i\epsilon/2), \epsilon/2) \cup \text{Ann}(0, \delta - r, \delta + r) \times D^{n-1}((0, \ldots, 0, -i\epsilon/2), \epsilon) =: H \subset \omega \); and

- \( \rho(z) < 0 \) for all \( z \in H \).

Therefore \( H \subset \Omega \). Now note that the set \( H' := D(0, \delta) \times D^{n-1}(0, \epsilon/2) \cup \text{Ann}(0, \delta - r, \delta + r) \times D^{n-1}(0, \epsilon) \) is a Reinhardt domain and it contains the point \( 0 \), whose each coordinate is \( 0 \). By Corollary 1.10 every holomorphic function on \( H' \) extends holomorphically to the set \( D(0, \delta) \times D^{n-1}(0, \epsilon) \). This implies that for every \( f \in H(\Omega) \), \( f|_H \) extends holomorphically to \( D(0, \delta) \times D^{n-1}((0, \ldots, 0, -i\epsilon/2), \epsilon) \). Since \( \Omega \) is a domain of holomorphy, we must have \( D(0, \delta) \times D^{n-1}((0, \ldots, 0, -i\epsilon/2), \epsilon) \subset \Omega \). In particular \( 0 \in \Omega \). This is a contradiction. Hence, our initial assumption cannot be sustained, and \( \Omega \) is Levi pseudoconvex. \( \square \)
Chapter 5

A schlichtness theorem for envelopes of holomorphy

In the previous chapters we discussed many equivalent conditions for a domain in $\mathbb{C}^n$ to be a domain of holomorphy. Also we saw that for a domain in $\mathbb{C}^n$ there exists a maximal domain of existence for every holomorphic function on the given domain, namely the envelope of holomorphy of the given domain. However we note that, the envelope of holomorphy is not necessarily sitting inside $\mathbb{C}^n$. Therefore it is apparently very difficult to obtain information about the envelope of holomorphy from information about the given domain in $\mathbb{C}^n$. By “information” we mean here any statement about the global geometry of the envelope of holomorphy (its local geometry is completely understood from the material in Chapter 3). In this chapter we present a result that gives us a very strong insight into the envelope of holomorphy based on information about the portion of the envelope that is “spread over” the given domain $\Omega$.

The theorem presented in this chapter is by Jupiter. We shall follow the notation used by Jupiter. To begin with we need some definitions.

**Definition 5.1.** Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $(\tilde{\Omega}, p, j)$ denote the envelope of holomorphy of $\Omega$. Let $X_1 \subset X_2 \subset \tilde{\Omega}$ be unramified domains spread over $\mathbb{C}^n$ such that $(X_j, p|_{X_j}, \mathbb{C}^n)$ are $H(\Omega)$-extensions of $\Omega$. We say that $X_2$ is schlicht over $X_1$ if $p|_{\tilde{\Omega}(p^{-1}(p(X_1)) \cap X_2)}$ is injective. We say that $X_1$ is schlicht over $\Omega$ if $X_1$ is schlicht over $j(\Omega)$. Finally, we say that $X_1$ is schlicht if $p|_{X_1}$ is injective.

**Definition 5.2** (Jupiter, [9]). Let $\Omega$ be a domain in $\mathbb{C}^2$. Let $z \in \tilde{\Omega}$. We say that $z$ can be reached from $\Omega$ by pushing discs if there is a neighbourhood $U$ of $z$ in $\tilde{\Omega}$ such that $p|_{U}$ is a biholomorphism and the following holds: there is a biholomorphism, $F$, of $D^2(0, 1)$ into $U$ such that

- $z \in F(D^2(0, 1))$; and
- $F(H) \Subset U \cap j(\Omega)$,

where $H$ is the Hartogs figure,

$$H := (\mathbb{D} \times \{\frac{1}{2} < |w| < 1\}) \cup (D(0, \frac{1}{2}) \times \mathbb{D}).$$
Remark 5.3. We should note that:

i) Without loss of generality we can take the neighbourhood $U$ to be biholomorphic to an open ball in $\mathbb{C}^2$.

ii) By Corollary 1.10 every holomorphic function on $F(H)$ has a single valued holomorphic extension into $F(D^2(0, 1))$.

iii) By rotating and scaling, if necessary, without loss of generality we can assume that if $z \in \overline{\Omega}$ can be reached from $\Omega$ by pushing discs, then $z \in F([0, 1] \times \mathbb{D})$.

In other words, we have a continuous family of holomorphic discs, contained $\overline{\Omega}$, such that the “bottom” disc is contained in $\Omega$, but the “top” disc does not lie in $\Omega$.

5.1 A schlichtness theorem

We now have sufficient background to state the main theorem of this chapter.

**Theorem 5.4** (Jupiter, [9]). Let $\Omega$ be a domain in $\mathbb{C}^2$, and $(\overline{\Omega}, p, j)$ be its envelope of holomorphy. If $\overline{\Omega}$ is schlicht over $\Omega$, then $\overline{\Omega}$ is schlicht.

Before we proceed to prove Theorem 5.4 we need two important lemmas.

Define $\Omega_n$ inductively as follows:

\[
\begin{align*}
\Omega_0 & := j(\Omega), \\
\Omega_{n+1} & := \{z \in \overline{\Omega} : z \text{ can be reached from } \Omega_n \text{ by pushing discs}\}.
\end{align*}
\]

Remark 5.5. Note that if $z \in \overline{\Omega}$ can be reached from $\Omega_n$ by pushing discs, then, by definition of pushing discs, every point in $F(D^2(0, 1)) \ni z$ can be reached from $\Omega_n$ by pushing discs.

Therefore for each $n \in \mathbb{Z}_+$, $\Omega_n$ is an open subset of $\overline{\Omega}$.

**Proposition 5.6.** Let $\Omega' := \bigcup_{n=0}^{\infty} \Omega_n$. Then $\Omega' = \overline{\Omega}$.

*Proof:* Let, if possible, $\Omega'$ be a proper subset of $\overline{\Omega}$. Then there exists $z \in \overline{\Omega}$ such that $z \in \partial \Omega'$ since $\overline{\Omega}$ is connected. Now, let $U$ be a neighbourhood of $z$ in $\overline{\Omega}$ such that $p|_U$ is a biholomorphism, $p(U)$ is an open ball in $\mathbb{C}^2$ and $U \cap \Omega'$ is connected. If $U \cap \Omega'$ is not a domain of holomorphy, then there exists a biholomorphism, $F$, of $D^2(0, 1)$ into $U$ such that

i) $F(H) \subset \Omega'$; and

ii) $F(D^2(0, 1))$ is not contained in $\Omega'$.

This follows from the fact that we can view $U \cap \Omega'$ as an open set in $\mathbb{C}^2$, since $p|_U$ is a biholomorphism, and from the argument in the final paragraph of our proof of Theorem 4.10.

Since $F(H)$ is contained in $\Omega'$, $F(H) \subset \Omega_n$ for some $n_0 \in \mathbb{N}$. Therefore, by definition of pushing discs, $F(D^2(0, 1)) = U \subset \Omega_{n_0}$, which contradicts the fact that $\Omega' \supset \Omega_{n_0}$.

Therefore by one of the many equivalent notions of domain of holomorphy, given in Chapter 4, we see that $\Omega'$ is a domain of holomorphy and $\Omega' \supset j(\Omega)$. This contradicts the fact that $\Omega' \subset \overline{\Omega}$. We conclude that $\Omega' = \overline{\Omega}$. \qed
To prove Theorem 5.4 we need the following lemmas. The hypothesis that $\Omega \subset \mathbb{C}^2$ is used in a major way in the following lemma. In fact, in [9] Jupiter constructs an example of an $\Omega \subset \mathbb{C}^3$ for which Lemma 5.7 fails, and shows that Theorem 5.4 can not be true for the latter $\Omega \subset \mathbb{C}^3$.

**Lemma 5.7.** Let us assume that for some $n \in \mathbb{N}$, $\Omega_n$ is schlicht. If $\Omega_{n+1}$ is schlicht over $\Omega_n$ then $\Omega_{n+1}$ is schlicht.

**Proof.** Assume, if possible, that $\Omega_{n+1}$ is not schlicht. Then there exists two points $z_1, z_2 \in \Omega_{n+1}$ such that $p(z_1) = p(z_2) = a$ (say). By the definition of $\Omega_{n+1}$ and the Remark 5.3, for each point $z_j$ there is a neighbourhood $U_j$ of $z_j$ in $\Omega$ and biholomorphic maps $F_j$ from $D^2(0, 1)$ into $U_j$ such that

1) $z_j \in F_j([0, 1] \times \mathbb{D});$ and

2) $F_j(H) \in U_j \cap \Omega_n.$

Define the set

$$T := ([0] \times \mathbb{D}) \cup ([0, 1] \times \partial \mathbb{D}).$$

We have $a \in p(F_1([0, 1] \times \mathbb{D})) \cap p(F_2([0, 1] \times \mathbb{D})).$ Let $X$ be the connected component of $p(F_1([0, 1] \times \mathbb{D})) \cap p(F_2([0, 1] \times \mathbb{D}))$ that contains $a.$ Note that as $p|_{U_j}$ is, by definition, a biholomorphism, $p(F_j([0, 1] \times \mathbb{D}))$ is a smooth manifold with boundary. Also, all of the previous statements remain true if we replace $F_j$ by a small perturbation — with respect to the $C^\infty(D^2(0, 1); U_j)$-metric — of $F_j,$ $j = 1, 2.$ Thus, by transversality, we may assume that $p(F_j([0, 1] \times \mathbb{D})), j = 1, 2,$ intersect in general position. Hence, we may assume that $X$ is a two-dimensional manifold. Moreover, for dimensional reasons, there exists a $t_j \in [0, 1]$ such that $M_j := (p|_{U_j})^{-1}(X) \cap F_j([t_j] \times \mathbb{D})$ is a one-dimensional real-analytic submanifold of $F_j([t_j] \times \mathbb{D}), j = 1, 2.$ By the properties of real-analytic submanifolds, $M_j \cap F_j([t_j] \times \partial \mathbb{D}) \neq \emptyset.$ Hence, there is a point $b \in X$ that lies either in $p(F_1(T))$ or in $p(F_2(T)).$ We then have that $b = p(F_j(\xi_j))$ for some $\xi_j \in T$ in the first case, or $b = p(F_2(\xi_2))$ for some $\xi_2 \in T$ in the second case. In other words, there exist points $\xi_j \in T$ such that in the first case $\beta_1 = F_1(\xi_1) \in \Omega_n$ and, in the second case $\beta_2 = F_2(\xi_2) \in \Omega_n.$

We now claim that both the points $\beta_1$ and $\beta_2$ cannot lie in $\Omega_n.$ If not, then since $\Omega_n$ is schlicht and $p(\beta_1) = p(\beta_2) = b,$ it follows that $\beta_1 = \beta_2.$ Now take any path in $X$ joining $b$ to $a.$ Then by the uniqueness of lifting (for unramified domains spread over open sets in Euclidean space), $z_1 = z_2.$ This is a contradiction.

Now we examine the case when $\beta_1 \in \Omega_n.$ Since $\beta_2 = F_2(\xi_2)$ does not belong to $\Omega_n,$ $\beta_1 \neq \beta_2.$ But $p(\beta_1) = p(\beta_2) = b.$ This implies that $\Omega_{n+1}$ is not schlicht over $\Omega_n$ which is a contradiction to our hypothesis.

The case when $b \in p(F_2(T))$ but is not contained in $p(F_1(T))$ can be dealt in similar way. Therefore $\Omega_{n+1}$ is schlicht. $\square$

**Lemma 5.8.** Assume that $\Omega_n$ is schlicht. If $\Omega_{n+1}$ is not schlicht over $\Omega_k,$ for some $k \leq n$ then $\Omega_{n+1}$ is not schlicht over $\Omega_{k-1}.$

**Proof.** Since $\Omega_{n+1}$ is not schlicht over $\Omega_k$ there exist $z_1 \in \Omega_{n+1}$ and $z_2 \in \Omega_k$ such that $p(z_1) = p(z_2) = a$ (say). By the definition of $\Omega_{n+1}$ and the Remark 5.3, for each point $z_j$ there is a neighbourhood $U_j$ of $z_j$ in $\Omega$ and biholomorphic maps $F_j$ from $D^2(0, 1)$ into $U_j$ such that
As in the proof of Lemma 5.7, let \( X \) be the connected component of \( p(F_1([0, 1] \times \mathbb{D})) \cap p(F_2([0, 1] \times \mathbb{D})) \) which contains \( a \). By a similar argument as in the proof of Lemma 5.7, \( X \) is a connected manifold and contains a point \( b \) which lies either in \( p(F_1(T)) \) or in \( p(F_2(T)) \). So, either \( b = p(F_1(\zeta_1)) \) for some \( \zeta_1 \) in \( T \) or \( b = p(F_2(\zeta_2)) \) for some \( \zeta_2 \) in \( T \).

Now, it is not possible that \( \beta_1 = F_1(\zeta_1) \) lies in \( \Omega_n \). If not, then since \( \beta_2 = F_2(\zeta_2) \) lies in \( \Omega_k \subset \Omega_n \) and \( \Omega_n \) is schlicht we shall have that \( \beta_1 = \beta_2 \). Similarly as in the previous proof, any path in \( X \) connecting \( a \) and \( b \) will give a contradiction.

Therefore \( \beta_1 \) lies in \( \Omega_{n+1} \) but does not lie in \( \Omega_n \) and \( \beta_2 \in F_2(T) \subset \Omega_{k-1} \). This means that \( \Omega_{n+1} \) is not schlicht over \( \Omega_{k-1} \).

\[ \Box \]

### 5.2 Proof of the main theorem

We now prove Theorem 5.4.

*Proof.* Suppose we knew that for each \( n \in \mathbb{N} \), \( \Omega_n \) is schlicht. Then since \( \Omega_n \supseteq \Omega_{n-1} \) for each \( n \in \mathbb{N} \) and \( \Omega = \bigcup_{n=0}^{\infty} \Omega_n \), \( p \) is injective on \( \Omega \). In other words \( \Omega \) is schlicht. Therefore it is sufficient to prove that each \( \Omega_n \) is schlicht.

Suppose \( \Omega_1 \) is not schlicht. By Lemma 5.7, \( \Omega_1 \) is not schlicht over \( \Omega_0 = j(\Omega) \). Therefore there exist two points \( z_1, z_2 \in \Omega_1 \) such that \( p(z_1) = p(z_2) \in \Omega \). This contradicts the fact that \( \Omega \) is schlicht over \( \Omega \) as \( \overline{\Omega} \supseteq \Omega_1 \). Therefore \( \Omega_1 \) is schlicht.

Let \( \Omega_n \) be schlicht. If \( \Omega_{n+1} \) is not schlicht then by Lemma 5.7, \( \Omega_{n+1} \) is not schlicht over \( \Omega_n \). By repeating Lemma 5.8 \( (n-1) \) times we get that \( \Omega_{n+1} \) is not schlicht over \( \Omega_0 \), which contradicts the fact that \( \Omega \) is schlicht over \( \Omega \).

Therefore if \( \Omega_n \) is schlicht then \( \Omega_{n+1} \) is schlicht. As \( \Omega_1 \) is schlicht, by induction we have \( \Omega_n \) is schlicht for every \( n \in \mathbb{Z}_+ \). \[ \Box \]
In this chapter, we explore another tool for obtaining information about the envelope of holomorphy from information about the given domain in $\mathbb{C}^n$.

In several recent papers on the subject of analytic continuation — see [1,4,5], for instance — one comes across versions of an informal lemma referred as “Kontinuitätssatz” or “the continuity principle”. The various conclusions deduced from it are obtainable from the following general principle:

\[\text{(*) Let } \Omega \text{ be a domain in } \mathbb{C}^n, n \geq 2, \text{ let } \{D_t\}_{t \in [0,1]} \text{ be an indexed family of smoothly bounded open sets in } \mathbb{C}, \text{ and let } \Psi_t : \overline{D_t} \to \mathbb{C}^{n-1} \text{ be maps such that } \Psi_t \in H(D_t) \cap C(\overline{D_t}) \text{ for each } t \in [0,1]. \text{ Assume that the } D_t \text{'s vary continuously with } t \text{ in some appropriate sense such that } \Gamma := \bigcup_{t \in [0,1]}(\overline{D_t} \times \{t\}) \text{ forms a “nice” compact body: such that } (\mathbb{C} \times (0,1)) \cap \partial \Gamma \text{ is a smooth submanifold with boundary, for instance. Also assume that } F : \Gamma \to \mathbb{C}^{n-1} \text{ given by } F(\zeta,t) := \Psi_t(\zeta) \text{ is continuous. Suppose we know that:}
\]

\[i) \quad \{(\zeta,\Psi_t(\zeta)) : \zeta \in \partial D_t \}\subset K, \text{ for some compact set } K \subset \Omega \text{ for each } t \in [0,1];
\]

\[ii) \quad \text{Graph}(\Psi_0) \subset \Omega \text{ and } \text{Graph}(\Psi_1) \not\subset \Omega.
\]

Then there exists a neighbourhood $V$ of $\text{Graph}(\Psi_1) \cup K$ and a neighbourhood $W$ of $K$, $K \subset W \subset \Omega \cap V$, such that for each $f \in H(\Omega)$ there exists a $G_f \in H(V)$ and $G_f|_W \equiv f|_W$.

Here, if $D$ is an open set then $H(D)$ denotes the class of holomorphic functions or maps on $D$.

To the best of my knowledge, there does not seem to be a proof of (\text{(*)}) in the above generality in the literature. If $\text{Graph}(\Psi_t) \cap \text{Graph}(\Psi_s) = \emptyset$ for $s \neq t$, then (\text{(*)}) is classical. If $D_t = D$, a fixed planar region, for each $t \in [0,1]$, then the above can be deduced by analytically continuing an $H(\Omega)$-germ (see the first page of Chapter 3 for a definition) along the path $t \mapsto (\zeta,\Psi_t(\zeta))$ for a fixed $\zeta \in D$. In more generality, but still assuming $\text{Graph}(\Psi_t) \cap \text{Graph}(\Psi_s) = \emptyset$ for $s \neq t$, (\text{(*)}) follows from Behnke’s work [2].

\[\text{When } \text{Graph}(\Psi_t) \cap \text{Graph}(\Psi_s) \neq \emptyset \text{ for some } t \neq s, \text{ we must worry about multivaluedness. This forces us to ask whether the maps } \text{id}_{\overline{D_t} \times \Psi_t} \text{ can be “lifted” into the envelope of holomorphy of } \Omega \text{ continuously in the parameter } t \in [0,1]. \text{ With } D_t = \mathbb{D}, \text{ the open unit disc in } \mathbb{C}, \text{ for all} \]

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$t \in [0, 1]$, and with maps more general than the form $\text{id}_{\mathbb{D}} \times \Psi_t$ (to be precise: continuously varying holomorphic immersions into $\mathbb{C}^n$), Jörice has shown that such “lifting” is possible [8, Lemma 1]. With some care, the conclusion of ($\ast$) can now be deduced in this special case — but it would be nice to have some record of the latter argument in the literature.

As for ($\ast$) in the above generality: it appears to be a folk theorem (also see the remark following Theorem 6.1). The aim of this chapter is to give a proof of ($\ast$) after making precise some of the loosely-stated assumptions in ($\ast$). The two theorems presented below are from the article [3], currently submitted for publication.

Some notation: if $S \subset \mathbb{C} \times [0, 1]$, we shall use $\mathcal{T}_S$ to denote the set $\{\zeta \in \mathbb{C} : (\zeta, t) \in S\}$.

**Theorem 6.1.** Let $\Omega$ be a domain in $\mathbb{C}^n$, $n \geq 2$. Let $K$ be a compact subset of $\Omega$. Let $\Gamma$ be a compact body in $\mathbb{C} \times [0, 1]$ such that $\partial_\ast \Gamma := (\mathbb{C} \times (0, 1)) \cap \partial \Gamma$ is a (not necessarily connected) $C^1$-smooth submanifold with boundary. Assume that the sets

$$D_t := \begin{cases} \mathcal{T}(\Gamma), & \text{if } t \in (0, 1), \\ \mathcal{T}(\Gamma) \cap (\mathbb{C} \setminus \mathcal{T}(\partial, \Gamma)), & \text{if } t = 0, 1, \end{cases}$$

are domains in $\mathbb{C}$ with $C^1$-smooth boundaries for each $t \in [0, 1]$, and that the set-valued function $[0, 1] \ni t \mapsto \mathcal{T}_t$ is continuous relative to the Hausdorff metric (on the space of compact subsets of $\mathbb{C}$). Suppose there is a continuous map $\Psi : \Gamma \to \mathbb{C}^n$ with the following properties:

1) For each $t \in [0, 1]$, $\Psi_t := \Psi(\cdot, t) \in H(D_t) \cap C(\mathcal{T}_t)$;

2) $\Psi_t(\partial D_t) \subseteq K$ for all $t \in [0, 1]$;

3) $\Psi_0(D_0) \subset \Omega$ and $\Psi_1(D_1) \not\subseteq \Omega$;

4) For each $t \in [0, 1]$, $\Psi_t$ is a continuous imbedding and $\Psi_t|_{D_t}$ is a holomorphic imbedding.

Then there exists a neighbourhood $V$ of $\mathcal{T}_1(\mathcal{T}(\Gamma)) \cup K$ and a neighbourhood $W$ of $K$ satisfying $K \Subset W \subset \Omega \cap V$ such that for any $f \in H(\Omega)$, there exists $G_f \in H(V)$ such that $G_f|_W \equiv f|_W$.

**Remark 6.2.** Theorem 6.1 would follow as a special case of a Kontinuitätssatz-type result by Chirka and Stout [6]. However, there are gaps in their proof. L. Nobel has shown in his thesis [11] that the Chirka–Stout Kontinuitätssatz is erroneous. Now, Theorem 6.1 does not directly follow from Nobel’s (unpublished) alternative to Chirka–Stout’s statement. Moreover, the approach of Chirka–Stout and Nobel requires much machinery, such as currents, volume inequalities, etc. Thus, an elementary proof in the set-up of Theorem 6.1 would be desirable.

In this proof, we shall continue with the notation introduced in Chapter 5. So, $(\Omega, p, j)$ will denote the envelope of holomorphy of $\Omega$. We recall: $p$ is the canonical local homeomorphism into $\mathbb{C}^n$ that gives $\Omega$ the structure of an unramified domain spread over $\mathbb{C}^n$ and $j : \Omega \leftrightarrow \Omega$ is an imbedding such that $p \circ j \equiv \text{id}_\Omega$. The only machinery that our proof of Theorem 6.1 requires is Thullen’s construction of the envelope of holomorphy of $\Omega$ — see Theorem 3.6 above. All other elements of our proof are elementary and developed from scratch.
Now, the usefulness of the Kontinuitätssatz is that it gives us information about \( p(\tilde{\Omega}) \). Generally, we have very little information about how \( \tilde{\Omega} \) looks. But, if we have explicit information about the pair \( (\Omega, K) \) occurring in Theorem 6.1, then we gain information about \( p(\tilde{\Omega}) \). In our proof of Theorem 6.1, an explicit description of \( V \), in terms of \( \Omega, K \) and \( \Psi_1(\tilde{D}_1) \), is obtained.

If we have slightly more information about \( (\Omega, p, j) \) than is available in Theorem 6.1, then it should be possible to relax some of the conditions on the family \( \{\Psi_t\}_{t \in [0,1]} \). For example: could we drop the requirement that \( \Psi_t, t \in [0,1] \), be imbeddings? This leads to our next theorem.

**Theorem 6.3.** Let \( \Omega \) be a domain in \( \mathbb{C}^2 \), and let \( (\tilde{\Omega}, p, j) \) denote the envelope of holomorphy of \( \Omega \). Let \( K \) be a compact subset of \( \Omega \). Let \( \Gamma \) be a compact body having exactly the same properties as in Theorem 6.1 and let the domains \( D_t, t \in [0,1] \), be the same as in Theorem 6.1. Suppose there is a continuous map \( \Psi : \Gamma \to \mathbb{C}^n \) having the properties (1), (2) and (3) stated in Theorem 6.1. Let us assume that \( p \) is injective on \( p^{-1}(\Omega) \). Then there exists an open set \( V \supset \Psi_1(\tilde{D}_1) \cup \Omega \) such that for each \( f \in H(\Omega) \), there exists \( G_f \in H(V) \) such that \( G_f|_{\Omega} \equiv f \).

Both theorems require a description of Thullen’s construction of the envelope of holomorphy. Salient features from Chapter 3 are recalled in Section 6.1. Also, we require a lifting result for the family \( \{\Psi_t\}_{t \in [0,1]} \). We state and proof this result in Section 6.2.

### 6.1 Recalling notations and terminology

Recall that Thullen’s construction depends on the sheaf of \( S \)-germs on \( \mathbb{C}^n \), denoted here as \( O(S) \). Precise definitions have been given in Chapter 3. The case that is important for us here is when \( S = H(\Omega) \).

We shall use the notation evolved in Chapter 3. Thus, throughout this paper, if \( (U, \phi_f : f \in H(\Omega)) \) is a representative of an \( H(\Omega) \)-germ at \( a \), we shall denote that germ as \( [U, \phi_f : f \in H(\Omega)]_a \). We shall denote the canonical extension of \( f \) to \( \tilde{\Omega}, f \in H(\Omega) \), as \( F_f \). For the reader’s convenience, let us also recapitulate/summarize:

1. \( \tilde{\Omega} \) is realised as the connected component of \( O(H(\Omega)) \) (i.e. the sheaf of \( H(\Omega) \)-germs of holomorphic functions) containing the (connected) set \( \{[\Omega, f : f \in H(\Omega)]_a : a \in \Omega \} \).

2. Given any \( f \in H(\Omega) \), the function \( F_f \) is defined on \( \tilde{\Omega} \) as follows. Each point \( x \in \tilde{\Omega} \) is an \( H(\Omega) \)-germ, so there exists a neighbourhood \( U \), containing \( a = p(x) \in \mathbb{C}^n \), such that \( x = [U, \phi_g : g \in H(\Omega)]_a \). Then \( F_f(x) := \phi_f(a) \).

Consider an unramified domain \( (X, p_0, \mathbb{C}^n) \) spread over \( \mathbb{C}^n \). Let \( a \in X \). Refer to Definition 3.18 for the definition of a polydisc of radius \( r \) about \( a \). We denote this object by \( P(a, r) \). The maximal polydisc about \( a \) is the union of all polydiscs about \( a \). As demonstrated in Section 3.2, the maximal polydisc is itself a polydisc about \( a \), and its radius

\[
r_0 = \sup\{r > 0 : P(a, r) \text{ is a polydisc about } a\}.
\]
Finally — see Definition 3.20 — the radius of the maximal polydisc about \(a\) is called the *distance of \(a\) from the boundary of \(X\) and is denoted by \(d_X(a)\). If \(A \subset X\), we set

\[
d_X(A) := \inf_{a \in A} d_X(a)
\]

### 6.2 A fundamental proposition

In this section, we shall prove a proposition that allows us to tackle both Theorem 6.1 and Theorem 6.3 within the same framework. We point out to the reader that an essential part of the proof of the proposition below depends on the structure of Thullen’s construction of the envelope of holomorphy \((\tilde{\Omega}, p, j)\). The other important fact that we use is the Cartan–Thullen characterization of a domain of holomorphy — see Theorem 3.26.

Recall that, given a compact \(K \subset X\), the holomorphically convex hull of \(K\) is the set

\[
\tilde{K}_X := \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in H(X)\}.
\]

We make one small notational point. In this section, if \(z = (z_1, \ldots, z_n) \in \mathbb{C}^n\), then

\[
||z|| := \max\{|z_1|, \ldots, |z_n|\}.
\]

**Proposition 6.4.** Let \(\Omega\) be a domain in \(\mathbb{C}^n\), \(n \geq 2\), and let \((\tilde{\Omega}, p, j)\) denote the envelope of holomorphy of \(\Omega\). Let \(K\) be compact subset of \(\Omega\). Let \(\Gamma\) be a compact body in \(\mathbb{C} \times [0, 1]\) having exactly the same properties as in Theorem 6.1. Let \(\Psi : \Gamma \to \mathbb{C}^n\) be a continuous map satisfying conditions (1), (2) and (3) from Theorem 6.1. Then, for each \(t \in [0, 1]\), there exists a continuous map \(\tilde{\Psi}_t : D_t \to \tilde{\Omega}\) such that \(p \circ \tilde{\Psi}_t \equiv \Psi_t\) and satisfies \(\tilde{\Psi}_t(\zeta) = [\Omega, f : f \in H(\Omega)]_{\Psi_t(\zeta)}\) for every \(\zeta \in \partial D_t\).

**Proof.** Define the set

\[
S := \{t \in [0, 1] : \exists \tilde{\Psi}_t : D_t \to \tilde{\Omega}, \tilde{\Psi}_t \in C(\overline{D}_t), p \circ \tilde{\Psi}_t \equiv \Psi_t\text{ and }\tilde{\Psi}_t(\zeta) = [\Omega, f : f \in H(\Omega)]_{\Psi_t(\zeta)}\text{ for all }\zeta \in \partial D_t\text{ and for all }t' \in [0, t]\}.
\]

We shall show that \(S\) is both open and closed in \([0, 1]\) and \(S \neq \emptyset\). To begin, define \(\tilde{\Psi}_0 : \overline{D}_0 \to \tilde{\Omega}\) by

\[
\tilde{\Psi}_0(\zeta) := [\Omega, f : f \in H(\Omega)]_{\Psi_0(\zeta)}\text{ for all }\zeta \in \overline{D}_0.
\]

Since \(\Psi_0(\overline{D}_0) \subset \Omega\), (6.1) is well-defined. Continuity is easy. Therefore \(0 \in S\).

Let \(\tau = \sup S\). By the definition of \(S\) we have that \([0, \tau) \subset S\). So to show that \(S\) is closed it is therefore enough to show that \(\tau \in S\). For this purpose, let us consider the envelope of holomorphy, \((\tilde{\Omega}, p, j)\). We recall an important aspect of the construction of \(\tilde{\Omega}\) (given at the beginning of Section 6.1). For each \(f \in H(\Omega)\), its canonical extension to \(\tilde{\Omega}\) is described as follows:
(**) For each point \(x \in \tilde{\Omega}\), which is an \(H(\Omega)\)-germ, let \(a = p(x) \in \mathbb{C}^n\) and let \(U\) be a neighbourhood of \(a\) such that \(x = [U, \phi_g : g \in H(\Omega)]\). Then \(F_f(x) = \phi_f(a) = \phi_f(p(x))\).

**Claim:** If \(t \in S\), then \(\tilde{\Psi}_t(\tilde{D}_t) \subset (K^*)_{\tilde{\Omega}}\) where \(K^* = j(K)\).

**Proof of claim:** Let \(F \in H(\Omega)\). As \(p \circ \tilde{\Psi}_t = \Psi_t\), both \(p\) and \(\Psi_t\) are holomorphic, and \(p\) is a local isomorphism (biholomorphism) we have \(\tilde{\Psi}_t \in H(D_t)\) (by the definition of holomorphicity of maps with values in an unramified domain spread over \(\mathbb{C}^n\)). Therefore \(F \circ \tilde{\Psi}_t \in H(D_t) \cap C(\tilde{D}_t)\).

Let \(\zeta \in D_t\). By the maximum modulus principle

\[
|F(\tilde{\Psi}_t(\zeta))| = |F \circ \tilde{\Psi}_t(\zeta)| \leq \sup_{\partial D_t} |F \circ \tilde{\Psi}_t| = \sup_{\tilde{\Psi}_t(\partial D_t)} |F| \leq \sup_{K^*} |F|, \tag{6.2}
\]

because \(\tilde{\Psi}_t(\partial D_t) \subset K^*\) (this is a consequence of property (i), stated in Section 6.1, of the envelope of holomorphy). Thus \(\tilde{\Psi}_t(\tilde{D}_t) \subset (K^*)_{\tilde{\Omega}}\). Hence the claim.

For any \(\zeta \in \tilde{D}_t\) and \(t \in S\), let us denote

\[
\tilde{\Psi}_t(\zeta) = [U^{t,\zeta}, \phi_f^{t,\zeta} : f \in H(\Omega)]_{\Psi_t(\zeta)} \tag{6.3}
\]

where \(U^{t,\zeta}\) is some neighbourhood of \(\Psi_t(\zeta)\) and \(\phi_f^{t,\zeta} \in H(U^{t,\zeta})\).

In what follows, we shall abbreviate \(d(\zeta)(K^*)\) to \(d(K^*)\). There exists an \(\varepsilon > 0\), a priori \(\varepsilon = \varepsilon(t, \zeta)\), such that all the \(H(\Omega)\)-germs in the definition below makes sense and the open set

\[
\mathcal{N}(\varepsilon, t, \zeta) := \{[U^{t,\zeta}, \phi_f^{t,\zeta} : f \in H(\Omega)] : a \in D^0(\Psi_t(\zeta), \varepsilon)\} \tag{6.4}
\]

is contained in \(\tilde{\Omega}\). As \(\tilde{\Omega}\) is a domain of holomorphy, it follows from the above claim and from the Cartan–Thullen Theorem that \(P(\Psi_t(\zeta), d(K^*)) \subset \tilde{\Omega}\) for all \(t \in S\) and for all \(\zeta \in \tilde{D}_t\). Here \(P(\Psi_t(\zeta), d(K^*))\) is as explained in Definition 3.18. It follows from the observations following Definition 3.18 that \(p|_{P(\Psi_t(\zeta), d(K^*))}\) is invertible. Thus, the functions

\[
\Phi_f^{t,\zeta} := F_f \circ (p|_{P(\Psi_t(\zeta), d(K^*))})^{-1} \tag{6.5}
\]

are holomorphic functions on \(D^0(\Psi_t(\zeta), d(K^*))\). Given our description (**)) above of the construction of \(F_f\), the above statement gives us the following equality of \(H(\Omega)\) germs:

\[
[U^{t,\zeta}, \phi_f^{t,\zeta} : f \in H(\Omega)]_{\Psi_t(\zeta)} = [D^0(\Psi_t(\zeta), d(K^*)), \Phi_f^{t,\zeta} : f \in H(\Omega)]_{\Psi_t(\zeta)}.
\]

We summarise this as the following:

**Fact 1:** For any \(t \in S\) and \(\zeta \in \tilde{D}_t\), \((D^0(\Psi_t(\zeta), d(K^*)), \Phi_f^{t,\zeta} : f \in H(\Omega))\) is a representative of the \(H(\Omega)\)-germ \(\tilde{\Psi}_t(\zeta)\), where \(\Phi_f^{t,\zeta}\) is the function given by (6.5).

Given this fact, the open set \(\mathcal{N}(\varepsilon, t, \zeta)\) makes sense for any number \(\varepsilon\) such that \(0 < \varepsilon \leq d(K^*)\), and is contained in \(\tilde{\Omega}\) for all \(\zeta \in \tilde{D}_t\), for all \(t \in S\). Furthermore, it will be understood that, for the remainder of this proof, \(U^{t,\zeta} = D^0(\Psi_t(\zeta), d(K^*))\) and \(\phi_f^{t,\zeta} = \Phi_f^{t,\zeta}\) whenever referring to the representation (6.3) for \(\tilde{\Psi}_t\) for \(t \in S\).
As $\Psi : \Gamma \to \mathbb{C}^n$ is a continuous map and $\Gamma$ is compact, there exists $\delta > 0$ such that for $(\zeta, t_1), (\eta, t_2) \in \Gamma$,

$$|(\zeta, t_1) - (\eta, t_2)| < \delta \Rightarrow ||\Psi_{t_1}(\zeta) - \Psi_{t_2}(\eta)|| < d(K^*),$$

(6.6)

where, for any $(\zeta, t) \in \mathbb{C} \times [0, 1], |(\zeta, t)| := \sqrt{|\zeta|^2 + t^2}$.

In the following argument $D(a, R)$ will denote the disc in $\mathbb{C}$ with centre $a$ and radius $R$. Given our hypotheses:

- By the smoothness of $\partial D_t$, for each $t \in [0, 1]$, there exists an $\epsilon^* := \epsilon^*(t) > 0$ such that for each $z \in \mathcal{A}_t := \{\zeta \in \mathbb{C} : \text{dist}[\zeta, \partial D_t] < \epsilon^*\}$ there is a unique closest point in $\partial D_t$, and such that each connected component of $\partial D_t$ is a strong deformation retract of the component of $\mathcal{A}_t$ containing it.

- By the properties of $\partial, \Gamma$, there exists a $\delta_* := \delta_*(t) > 0$ such that $\partial, \Gamma$ divides each connected component of $\mathcal{A}_t \times ([0, 1] \cap [t - \delta_*, t + \delta_*])$ into two connected components.

It now follows from an elementary argument, using compactness of $[0, 1]$, that there exists a constant $r$, $0 < r \leq \delta$, such that for each $t \in [0, 1]$ and each $\zeta \in \overline{D}_t$, $D(\zeta, r) \cap \overline{D}_t$ is connected. Let us now fix a $t \in S$. By continuity, there exists a constant $\delta_t$, $0 < \delta_t \leq r$, depending only on $t$, such that

$$\eta \in D(\zeta, \delta_t) \cap \overline{D}_t \Rightarrow \overline{\Psi}_{t}(\eta) \in N(d(K^*), t, \zeta).$$

Therefore

$$\eta \in D(\zeta, \delta_t) \cap \overline{D}_t \Rightarrow \overline{\Psi}_{t}(\eta) = \{U^t_{\zeta}, \phi^t_{\zeta} : f \in H(\Omega)\}_{\Psi_t(\eta)}. \quad (6.7)$$

Let us now fix $\zeta \in \overline{D}_t$. Set $A(t, \zeta) := D(\zeta, r) \cap \overline{D}_t$. Due to connectedness, for any $x \in A(t, \zeta)$, there is a finite chain of discs $\Delta_1, \Delta_2, \ldots, \Delta_{N(x)}$ of radius $\delta_t$ and points $\zeta = y_0, y_1, \ldots, y_{N(x)} = x$ such that

$$y_j = \text{centre of } \Delta_{j+1}, \quad y_{j+1} \in \Delta_{j+1} \text{ for } j = 0, 1, \ldots, N(x) - 1.$$ 

Suppose we have been able to show that for any $x \in A(t, \zeta)$ that can be linked to $\zeta$ by a chain of at most $M$ discs in the above manner, we have

$$\overline{\Psi}_t(x) = \{U^t_{\zeta}, \phi^t_{\zeta} : f \in H(\Omega)\}_{\Psi_t(x)}.$$

Now let $z \in A(t, \zeta)$ be a point that is linked to $\zeta$ by a chain $\Delta_1, \Delta_2, \ldots, \Delta_{M+1}$ of discs of radius $\delta_t$. Let $y_M$ be the centre of $\Delta_{M+1}$. By our inductive hypothesis

$$\overline{\Psi}_t(y_M) = \{U^t_{\zeta}, \phi^t_{\zeta} : f \in H(\Omega)\}_{\Psi_t(y_M)}.$$

By the representation (6.3) and the fact that $U := U^t_{\zeta} \cap U^{t_M}$ is convex,

$$\phi^t_{\zeta}|_U \equiv \phi^{t_M}|_U \text{ for all } f \in H(\Omega).$$

(6.8)
Now applying (6.7) to the pair \((y_M, z)\), we get
\[
\tilde{\Psi}_r(z) = \{U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)\}_{\Psi_r(z)}
= \{U, \phi_f |^{1,0}_m : f \in H(\Omega)\}_{\Psi_r(z)},
\]
which follows from (6.8) and the fact that, as \(|z - y_M| < \delta, \Psi_r(z) \in U\). From the last equality, we get \(\tilde{\Psi}_r(z) = \{U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)\}_{\Psi_r(z)}\). Given (6.7), mathematical induction tells us that we have actually established the following:

**Fact 2:** For any \(t \in S\) and \(\zeta \in \overline{D}_t\),
\[
\phi_f |^{1,0}_m \equiv \phi_f |^{1,0}_m \quad \text{for all } f \in H(\Omega),
\]
and for any \(\eta \in \overline{D}_t\) such that \(|\eta - \zeta| < r\), where \(X(\zeta, \eta) = U |^{1,0}_m \cap U |^{1,0}_m\).

Now we have that \(\overline{D}_1 \subset [0,1] \) is a continuous family (hence a uniformly continuous family) with respect to the Hausdorff metric. From uniform continuity, the fact that \([0, r] \subset S\), and from the properties of \(\partial, \Gamma\), we can find a \(t_0 \in S\) such that \((\tau - t_0) \leq \gamma/4\) and such that
\[
\overline{D}_t \subset B(\overline{D}_{t_0}, \gamma/4), \quad \partial D_t \subset B(\partial D_{t_0}, \gamma/4),
\]
where, given a set \(A \subset \mathbb{C}\) and \(C > 0\), \(B(A, C) := \cup_{\zeta \in A} D(\zeta, C)\). Let \(\zeta \in \overline{D}_t\). Then there exists \(x(\zeta) \in \overline{D}_{t_0}\) such that \(|\zeta - x(\zeta)| < \gamma/4\). Define
\[
\tilde{\Psi}_r(\zeta) := \{U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)\}_{\Psi_r(\zeta)}.
\]
As \(r \leq \delta\), it follows from Fact 1 and (6.6) that \(\tilde{\Psi}_r(\zeta) \in \overline{\Omega}\).

**Claim:** The function \(\tilde{\Psi}_r : \overline{D}_t \rightarrow \overline{\Omega}\) is well-defined and continuous.

**Proof of claim:** Let \(\zeta \in \overline{D}_t\). Suppose \(x(\zeta), y(\zeta)\) are two different points in \(\overline{D}_{t_0}\) such that \(|\zeta - x(\zeta)| < \gamma/4\), \(|\zeta - y(\zeta)| < \gamma/4\). Then \(|x(\zeta) - y(\zeta)| < \gamma/2\). As \((\tau - t_0) \leq \gamma/4\), it follows from (6.6) that \(\Psi_r(x(\zeta)) \in U |^{1,0}_m \cap U |^{1,0}_m\). Therefore, by (6.9)
\[
[U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)]_{\Psi_r(z)} = [U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)]_{\Psi_r(z)}.
\]
Therefore \(\tilde{\Psi}_r\) is well-defined.

We now pick a \(\zeta \in \overline{D}_t\) and fix it. By the construction of the topology on \(\overline{\Omega}\), given any neighbourhood \(G\) of \(\tilde{\Psi}_r(\zeta)\) in \(\overline{\Omega}\), there exists \(\epsilon \in (0, d(K^*))\), sufficiently small, such that \(N(\epsilon, \tau, \zeta) \subset G\). Now let \(0 < \epsilon < d(K^*)\). Since \(\Psi_r : \overline{D}_t \rightarrow \mathbb{C}\) is continuous there exists \(\sigma = \sigma(\epsilon)\), with \(\sigma(\epsilon) \in (0, \gamma/4)\), such that if \(|\eta - \zeta| < \sigma, \eta \in \overline{D}_t\), then \(\Psi_r(\eta) \in D(\Psi_r(\zeta), \epsilon)\). Let \(\eta \in \overline{D}_t\) such that \(|\eta - \zeta| < \sigma\). There exists \(x(\eta) \in \overline{D}_{t_0}\) such that \(|\eta - x(\eta)| < \gamma/4\). Therefore \(|x(\zeta) - x(\eta)| < \gamma/4\). Recall that \((\tau - t_0) \leq \gamma/4\). Therefore, applying (6.6) and (6.9) we get \([U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)]_{\Psi_r(\eta)} = [U |^{1,0}_m, \phi_f |^{1,0}_m : f \in H(\Omega)]_{\Psi_r(\eta)}\). Therefore \(\tilde{\Psi}_r(\eta) \in N(\epsilon, \tau, \zeta)\) whenever \(|\zeta - \eta| < \sigma\) and \(\eta \in \overline{D}_t\).

By the remarks at the beginning of this paragraph, \(\tilde{\Psi}_r\) is continuous at \(\zeta\). As \(\zeta\) is an arbitrary point of \(\overline{D}_t\) we have that \(\tilde{\Psi}_r\) is continuous. Hence the claim.
By definition of $\tilde{\Psi}$, $p \circ \tilde{\Psi} \equiv \Psi$. For all $x \in \partial D_0$, $\tilde{\Psi}_{t_0}(x) = [\Omega, f : f \in H(\Omega)]_{\Psi_{t_0}(x)}$. Since $\partial D_\tau \subset B(\partial D_0, \epsilon/4)$, for any $\zeta \in \partial D_\tau$, there exists an $x(\zeta) \in \partial D_0$ such that $\zeta \in D(x(\zeta), \epsilon/4)$. By the argument in the first part of the previous claim, $\tilde{\Psi}_\tau(\zeta) = [\Omega, f : f \in H(\Omega)]_{\Psi_{t}(\zeta)}$ for all $\zeta \in \partial D_\tau$. This implies that $\tau \in S$.

We have established that $S$ is closed.

The above method for showing that $\tau \in S$ tells us more. As $[0, 1] \ni t \mapsto \overline{D}_t$ is uniformly continuous relative to the Hausdorff metric, and as $\partial, \Gamma$ is smooth, there exists a constant $\delta^* > 0$ such that

$$|t - s| < \delta^* \Rightarrow \mathcal{H}(\overline{D}_t, \overline{D}_s) < \epsilon/4,$$

$$0 < (t - s) < \delta^* \Rightarrow \partial D_t \subset B(\partial D_s, \epsilon/4),$$

where $\mathcal{H}$ denotes the Hausdorff metric and $\epsilon$ is exactly as fixed above. Let $\delta^* := \min(\delta^*, \epsilon/4)$. The same argument, with appropriate replacements where necessary, shows that if $t_0 \in S$, then $\tau \in S$ for each $\tau \in [t_0, \min(1, t_0 + \delta^*))$. Therefore $S$ is both open and closed. We have shown that $0 \in S$. Hence $S = [0, 1]$, which completes the proof. 

6.3 The proof of the main theorems

The proof of Theorem 6.1. By Proposition 6.4 we have that there exists $\tilde{\Psi}_1 : \overline{D}_1 \to \overline{\Omega}$, a continuous map such that $p \circ \tilde{\Psi}_1 \equiv \Psi_1$ and $\tilde{\Psi}_1(\partial D_1) \subset K^\ast$. The condition $p \circ \tilde{\Psi}_1 \equiv \Psi_1$ actually implies that $\tilde{\Psi}_1 \in H(D_1)$, by the definition of holomorphy of maps with values in an unramified domain spread over $\mathbb{C}^n$. Thus, as argued in the proof of Proposition 6.4, $\tilde{\Psi}_1(\overline{D}_1) \subset (K^\ast)_{\overline{\Omega}}$. Therefore, by Theorem 3.26, $P(\tilde{\Psi}_1(\zeta), d(K^\ast)) \subset \Omega$ for all $\zeta \in \overline{D}_1$. Let us define for any $r > 0$,

$$\omega(r) := \{\zeta \in D_1 : \text{dist}[\zeta, \partial D_1] > r\}.$$

Claim: There exists $\epsilon_0 \in (0, d_\Omega(K)/2 \rangle$ such that if $\zeta, \eta \in \overline{D}_1$, and $D^n(\Psi_1(\zeta), \epsilon_0) \cap D^n(\Psi_1(\eta), \epsilon_0) \neq \emptyset$, then

either $D^n(\Psi_1(\zeta), \epsilon_0) \cup D^n(\Psi_1(\eta), \epsilon_0) \subseteq \Omega,$

or $D^n(\Psi_1(\zeta), 2\epsilon_0) \cap D^n(\Psi_1(\eta), 2\epsilon_0) \cap \Psi_1(\overline{D}_1) \neq \emptyset$ and

$$D^n(\Psi_1(x), 2\epsilon_0) \cap \Psi_1(\hat{D}_1), \ x = \zeta, \eta, \text{ is path-connected.} \quad (6.10)$$

Proof of claim: $K$ is a compact subset of $\Omega$. Therefore there exists an $\epsilon \in (0, d_\Omega(K))$ such that $D^n(\zeta, \epsilon) \subseteq \Omega$ for all $\zeta \in K$. Therefore, as $\Psi_1(\partial D_1) \subseteq K$ and $\Psi_1$ is continuous on $\overline{D}_1$, which is a compact set, there exists $\delta > 0$ such that $\Psi_1(\omega(2\delta))^\circ \subseteq \cup_{z \in K} D^n(z, \epsilon/4)$. By construction we see that for any $\epsilon \in (0, \epsilon/4)$, if $\zeta \in \overline{D}_1 \setminus \omega(2\delta)$ and $\eta \in \overline{D}_1$, then

$$D^n(\Psi_1(\zeta), \epsilon) \cap D^n(\Psi_1(\eta), \epsilon) \neq \emptyset \implies D^n(\Psi_1(\zeta), \epsilon) \cup D^n(\Psi_1(\eta), \epsilon) \subseteq \Omega. \quad (6.11)$$

The above statement remains true when $\zeta$ and $\eta$ are interchanged.
Let us view $\mathbb{C}^n$ as a Hermitian manifold with $T(\mathbb{C}^n)$ as the holomorphic tangent space equipped with the standard Hermitian inner product $\langle \cdot , \cdot \rangle_{std}$ on each $T_p\mathbb{C}^n$, $p \in \mathbb{C}^n$. Let

$$N_{\Psi_1} := \text{the normal bundle of } \Psi_1(D_1) \text{ with respect to } \langle \cdot , \cdot \rangle_{std}.$$ 

Let $\pi$ denote the bundle projection. It is well known that for any relatively compact subdomain $\Delta \Subset D_1$, there exists $r_\Delta > 0$, such that, if we define:

$$N(\Psi_1, r_\Delta) := \bigcup_{p \in \Psi_1(\Delta)} \{ v \in N_{\Psi_1} \mid_p : \langle v, v \rangle_{std} < r_\Delta^2 \},$$

then the map $\Theta(v) := \pi(v) + v$ is a holomorphic imbedding of $N(\Psi_1, r_\Delta)$ into $\mathbb{C}^n$.

Write $\Delta := \omega(\theta/2)$ and $\omega := \omega(\delta)$. Since $\Psi_1(\omega)$ is a compact subset of $\mathbb{C}^n$, there exists an $\epsilon_0 \in (0, \min(d_\Omega(K)/2, \ell/4))$ so small that:

- $D^n(\Psi_1(z), 2\epsilon_0) \cap \Psi_1(\overline{D}_1)$ is path-connected for all $z \in \Psi_1(\Delta)$;
- $D^n(z, 2\epsilon_0) \cap \Psi_1(\Delta) \Subset \Psi_1(\omega)$ (with respect to the relative topology on $\Psi_1(\Delta)$) for all $z \in \Psi_1(\omega)$;
- $D^n(z, \epsilon_0) \subseteq \Theta(N(\Psi_1, r_\Delta))$ for all $z \in \Psi_1(\omega)$; and
- $\pi \circ \Theta^{-1}(z) \in D^n(p, 2\epsilon_0) \cap \Psi_1(\Delta)$ for all $z \in D^n(p, \epsilon_0)$ and for all $p \in \Psi_1(\omega)$

(where we understand that $\Theta : N(\Psi_1, r_\Delta) \to \mathbb{C}^n$).

Now suppose $\zeta \neq \eta \in \Psi_1(\omega)$ and $D^n(\zeta, \epsilon_0) \cap D^n(\eta, \epsilon_0) \neq \emptyset$. Let $w \in D^n(\zeta, \epsilon_0) \cap D^n(\eta, \epsilon_0)$. Then, as $\Theta$ is an imbedding, there is a unique $v \in N(\Psi_1, r_\Delta)$ such that $\Theta(v) = w$. By our construction $\pi(v) = \pi \circ \Theta^{-1}(w) \in D^n(\zeta, 2\epsilon_0) \cap D^n(\eta, 2\epsilon_0) \cap \Psi_1(\Delta)$. Combining this with the statement culminating in (6.11), we have the claim.

Write:

$$V := \left( \bigcup_{\zeta \in D_1} D^n(\Psi_1(\zeta), \epsilon_0) \right) \cup \left( \bigcup_{z \in K} D^n(z, \epsilon_0) \right), \quad W := \bigcup_{z \in K} D^n(z, \epsilon_0).$$

Clearly, $W \subset V \cap \Omega$ as $\epsilon_0 < d_\Omega(K)$ and $V \not\subseteq \Omega$ as $\Psi_1(\overline{D}_1) \subseteq V$. Let us use the notation (6.3) to represent $\Psi_1(\zeta)$ (which is an $H(\Omega)$- germ). We have established in the proof of Proposition 6.4 — see Fact 1 stated in Section 6.2 — that there is a representative $(U^1, \phi^f : f \in H(\Omega))$ of $\Psi_1(\zeta)$ such that $U^1 = D^n(\Psi_1(\zeta), d(K^*)).$ Given any $f \in H(\Omega)$, define $G_f : V \to \mathbb{C}$ by

$$G_f(z) := \begin{cases} \phi^f(z), & \text{if } z \in D^n(\Psi_1(\zeta), \epsilon_0), \\ f(z), & \text{if } z \in D^n(\eta, \epsilon_0) \text{ for some } \eta \in K \setminus \Psi_1(\overline{D}_1). \end{cases}$$

Here, for simplicity of notation, we write $\phi^\zeta_f := \phi^{U^1}_{f^\zeta}$.

We shall prove that the function $G_f$ is well-defined on $V$. To start with, let us assume that for some $z \in V$, $z \in D^n(\Psi_1(\zeta), \epsilon_0) \cap D^n(\Psi_1(\eta), \epsilon_0)$, where $\zeta \neq \eta$. By the above claim, there are two possible outcomes. If $D^n(\Psi_1(\zeta), \epsilon_0) \cup D^n(\Psi_1(\eta), \epsilon_0) \subseteq \Omega$, then $\phi^\zeta_f = f = \phi^\eta_f$ by Proposition 6.4.
Therefore, we must focus on the outcome (6.10) given by the last claim. In this situation, there exists \( \tau \in D_1 \) such that \( \Psi_1(\tau) \in D^u(\Psi_1(\zeta), 2\epsilon_0) \cap D^u(\Psi_1(\eta), 2\epsilon_0) \). The same argument that leads to Fact 2 stated in the proof of Proposition 6.4 can be used to show the following analogue of Fact 2:

**Fact 2':** For any \( t \in [0, 1] \) and \( \zeta \in D_1 \), if \( A_\zeta := \) the path-component of \( \Psi_t^{-1}(U^{1,\zeta}) \) containing \( \zeta \), then

\[
\phi_f^{t,\zeta}|_{X(\zeta, \eta)} \equiv \phi_f^{t,\eta}|_{X(\zeta, \eta)} \quad \text{for all } f \in H(\Omega) \text{ and for all } \eta \in A_\zeta,
\]

where \( X(\zeta, \eta) = U^{1,\zeta} \cap U^{1,\eta} \).

Since \( 2\epsilon_0 < d_Ω(\kappa) \leq d(K^*) \), \( \Psi_1(\tau) \in U^{1,\tau} \), \( x = \zeta, \eta \). Furthermore, under the condition (6.10), we see that

\( \tau \) lies in the connected component of \( \Psi_1^{-1}(U^{1,\tau}) \) containing \( x, x = \zeta, \eta \),

since \( \Psi_1 \) is an imbedding. From Fact 2', we deduce that

\[
\phi_f^{t}|_{U^{1,\tau} \cap U^{1,\tau}} \equiv \phi_f^{t}|_{U^{1,\tau} \cap U^{1,\tau}}, \quad \phi_f^{t}|_{U^{1,\tau} \cap U^{1,\tau}} \equiv \phi_f^{t}|_{U^{1,\tau} \cap U^{1,\tau}},
\]

for all \( f \in H(\Omega) \). Therefore there is a neighbourhood \( N \) of \( \Psi_1(\tau) \) such that

\[
\phi_f^{t}|_{N} \equiv \phi_f^{t}|_{N}. \tag{6.12}
\]

As \( X := D^u(\Psi_1(\zeta), 2\epsilon_0) \cap D^u(\Psi_1(\eta), 2\epsilon_0) \) is connected, by (6.12) we have \( \phi_f^{t}|_{X} \equiv \phi_f^{t}|_{X} \). Therefore \( \phi_f^{t}(z) = \phi_f^{t}(z) \).

Lastly, we consider the case when \( z \in D^u(\Psi_1(\zeta), \epsilon_0) \cap D^u(\theta, \epsilon_0) \), where \( \zeta \in \overline{D}_1 \) and \( \theta \in K \). In this case, the argument for well-definedness is exactly as presented at the beginning of the last paragraph. This completes the proof of well-definedness.

By the construction of \( G_f \) on \( V \), we have, given that \( z \in D^u(\theta, \epsilon_0) \) for some \( \theta \in K \), \( G_f(z) = f(z) \). Therefore \( G_f|_{W} \equiv f|_{W} \). Finally, by the fact that holomorphicity is a local property, we conclude that \( G_f \in H(V) \). \( \square \)

We now turn to the proof of Theorem 6.3. For this purpose, we will use Theorem 5.4 by Jupiter.

**The proof of Theorem 6.3.** By Theorem 5.4, \( p : \overline{\Omega} \to \mathbb{C}^n \) is an injective map. As \( p \) is a local biholomorphism, \( p(\overline{\Omega}) \) is an open connected set in \( \mathbb{C}^n \) and \( p^{-1} \) is a holomorphic map on it. Let us write

\[
V := \bigcup_{\zeta \in \Psi_1(\overline{D}_1) \setminus K} D^u(z, d_\Omega(K)).
\]

Since \( p \circ \Psi_1 \equiv \Psi_1 \) and \( \Psi_1(\overline{D}_1) \subset \overline{\Omega} \), we have \( V \not\subset \Omega \). As \( \Psi_1(\overline{D}_1) \not\subset \Omega \).

As \( d_\Omega(K) \leq d(K^*) \) and \( \Psi_1(\overline{D}_1) \subset (K^*)_{\overline{\Omega}} \), the Cartan–Thullen Theorem gives \( V \subset p(\overline{\Omega}) \). Thus, if for any \( f \in H(\Omega) \), we define \( G_f : V \to \mathbb{C} \) by

\[
G_f := F_f \circ p^{-1},
\]

then \( G_f \in H(V) \) and \( G_f|_\Omega \equiv f \). \( \square \)
Remark 6.5. Note that we could have taken $V = p(\tilde{\Omega})$ in the above proof. But we want to highlight the fact that the Kontinuitätssatz is a means of deducing information about $p(\tilde{\Omega})$ when very little is known about the geometry of $\tilde{\Omega}$. The spirit of Theorem 6.3 is that, while we know very little about the finer geometric properties of $\tilde{\Omega}$, this theorem shows what we can learn about $p(\tilde{\Omega})$ if we have a little extra information about $(\tilde{\Omega}, p, j)$. In using both Theorem 6.3 and Theorem 6.1, it is understood that we have good information about the pair $(\Omega, K)$. It is in keeping with this spirit that we chose $V$ as above.
Bibliography


