Application of UQ and stochastic models to studying El Niño-Southern Oscillation

Instructor: Nan Chen

Department of Mathematics
University of Wisconsin-Madison

Graduate Course, MATH 801, Spring 2020
MWF 8:50AM – 9:40AM, B239 Van Vleck Hall
El Niño is a climate pattern that includes the interactions between
1. atmosphere, 2. ocean, 3. sea surface temperature (SST).

El Niño & La Niña are the opposite phases of El Niño-Southern Oscillation (ENSO).
The anomalous climate patterns in the equatorial Pacific affect global climate through teleconnections.
The Simplest ENSO Theory

A discharge-recharge model (Jin 1997),

\[
\frac{dT}{dt} = -\lambda T + \omega H, \\
\frac{dH}{dt} = -\omega T,
\]

where \( T \) represents the eastern Pacific SST anomalies and \( H \) is the anomalous western Pacific (warm pool) heat content.
Two Different Super El Niño Events

- Most strong El Niños have similar formation mechanisms as the 1997-1998 event.
- However, the super El Niño during 2014-2016 is quite different.

What happens to the 2014-2016 event?

1997-1998

- Zonal Winds
- SST
- Thermocline depth
Two Different Super El Niño Events

- Most strong El Niños have similar formation mechanisms as the 1997-1998 event.
- However, the super El Niño during 2014-2016 is quite different.

What happens to the 2014-2016 event?
Two Different Super El Niño Events

- Most strong El Niños have similar formation mechanisms as the 1997-1998 event.
- However, the super El Niño during 2014-2016 is quite different.

What happens to the 2014-2016 event?

**2014 Spring:** SST and ocean’s response similar to the 97-98 event.

**2014 Fall:** A weak El Niño and then it seemed to disappear.

**2015:** The eastward winds strengthened, and the already warm water in the central Pacific helped push things along: a super El Niño appeared!
Goal of this work.

- Developing a simple modeling framework for El Niño.
- Studying the mechanisms of different super El Niños.
The Starting Coupled Model: Deterministic, Linear and Stable

Atmosphere
\[-yv - \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q/(1 - \bar{Q})\]
\[u, v: \text{winds} \quad \theta: \text{temperature} \quad E_q = \alpha T: \text{latent heat}\]

Ocean
\[\partial_T U - c_1 YV + c_1 \partial_x H = c_1 \tau_x\]
\[YU + \partial_Y H = 0\]
\[\partial_T H + c_1 (\partial_x U + \partial_Y V) = 0\]
\[U, V: \text{ocean current} \quad H: \text{thermocline depth} \quad \tau_x = \gamma u: \text{wind stress}\]

SST
\[\partial_T T = -c_1 \zeta E_q + c_1 \eta H\]
\[T: \text{sea surface temperature} \quad \eta: \text{thermocline feedback} \quad (\eta \text{ stronger in eastern Pacific})\]

(zonal \(\leftrightarrow\): east-west; meridional \(\updownarrow\): north-south)
The Starting Coupled Model: Deterministic, Linear and Stable

Atmosphere
\[-yv - \partial_x \theta = 0\]
\[yu - \partial_y \theta = 0\]
\[-(\partial_x u + \partial_y v) = E_q/(1 - \overline{Q})\]
\[u, v: \text{winds}\]
\[\theta: \text{temperature}\]
\[E_q = \alpha T: \text{latent heat}\]

Ocean
\[\partial_T U - c_1 YV + c_1 \partial_x H = c_1 \tau_x\]
\[YU + \partial_Y H = 0\]
\[\partial_T H + c_1(\partial_x U + \partial_Y V) = 0\]
\[U, V: \text{ocean current}\]
\[H: \text{thermocline depth}\]
\[\tau_x = \gamma u: \text{wind stress}\]

SST
\[\partial_T T = -c_1 \zeta E_q + c_1 \eta H\]
\[T: \text{sea surface temperature}\]
\[\eta: \text{thermocline feedback}\]

- Different from nonlinear models that use internal instability to trigger the ENSO cycles.
- Non-dissipative atmosphere consistent with the skeleton model of Madden-Julian Oscillation (Majda and Stechmann 2009, 2011); suitable to describe the dynamics of the Walker circulation
- Different meridional axis $y$ and $Y$ due to different Rossby radius in atmosphere and ocean
- Allowing a systematic meridional decomposition of the system into the well-known parabolic cylinder functions, keeping the system low-dimensional

(zonal $\leftrightarrow$: east-west  meridional $\uparrow\downarrow$: north-south)
Reduction: Meridional Truncation.
(zonal (↔): east-west; meridional (↕): north-south)

Atmosphere: a truncation of heat sources to the first parabolic cylinder function $\phi_0$.

Excite \[
\{ \begin{align*}
\text{atmospheric Kelvin wave } & K_A \\
\text{first atmospheric Rossby wave } & R_A
\end{align*} \]

Ocean: a truncation of zonal wind stress forcing to $\psi_0$.

Excite \[
\{ \begin{align*}
\text{oceanic Kelvin wave } & K_O \\
\text{first oceanic Rossby wave } & R_O
\end{align*} \]

SST: Same as ocean $\psi_0$. 

Meridional Bases

\[
\begin{align*}
\text{Atm } \phi_0 & \ldots \text{Atm } \phi_2 \\
\text{Ocn } \psi_0 & \ldots \text{Ocn } \psi_2
\end{align*}
\]
Meridional truncation \((x, y, \tau) \rightarrow (x, \tau)\)

<table>
<thead>
<tr>
<th>Original</th>
<th>Truncated to (\phi_0(y)) and (\psi_0(y))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Atmosphere</strong></td>
<td>- (yv - \partial_x \theta = 0)</td>
</tr>
<tr>
<td></td>
<td>- (yu - \partial_y \theta = 0)</td>
</tr>
<tr>
<td></td>
<td>- ((\partial_x u + \partial_y v) = E_q/(1 - \bar{Q}))</td>
</tr>
<tr>
<td><strong>Ocean</strong></td>
<td>- (\partial_\tau U - c_1 YV + c_1 \partial_x H = c_1 \tau_x)</td>
</tr>
<tr>
<td></td>
<td>- (YU + \partial_Y H = 0)</td>
</tr>
<tr>
<td></td>
<td>- (\partial_\tau H + c_1(\partial_x U + \partial_Y V) = 0)</td>
</tr>
<tr>
<td><strong>SST</strong></td>
<td>- (\partial_\tau T = -c_1 \zeta E_q + c_1 \eta H)</td>
</tr>
</tbody>
</table>

Reconstructed variables:

- \(u = (K_A - R_A)\phi_0 + (R_A/\sqrt{2})\phi_2\)
- \(\theta = -(K_A + R_A)\phi_0 - (R_A/\sqrt{2})\phi_2\)
- \(U = (K_O - R_O)\psi_0 + (R_O/\sqrt{2})\psi_2\)
- \(H = (K_O + R_O)\psi_0 + (R_O/\sqrt{2})\psi_2\)

Meridional Bases

\[y (1000\text{km})\]

\[\text{Atm } \phi_0 \text{ Atm } \phi_2 \text{ Ocn } \psi_0 \text{ Ocn } \psi_2\]
Method of characteristics is applied to derive the meridional truncated equations.

A good reference to follow is the following book
Introduction to PDEs and Waves for the Atmosphere and Ocean, 2003, by Andrew J. Majda, Chapter 9

Other useful references for technical details are:
Stechmann and Majda, *Monthly Weather Review*, 2015, Appendix,
Biello and Majda, *Dynamics of Atmospheres and Oceans*, 2006, Appendix c.4
Numerical solver of the coupled model

We solve the meridionally truncated model.

<table>
<thead>
<tr>
<th>Original</th>
<th>Truncated to $\phi_0(y)$ and $\psi_0(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Atmosphere</strong></td>
<td>$-yv - \partial_x \theta = 0$</td>
</tr>
<tr>
<td></td>
<td>$yu - \partial_y \theta = 0$</td>
</tr>
<tr>
<td></td>
<td>$-(\partial_x u + \partial_y v) = E_q/(1 - \bar{Q})$</td>
</tr>
<tr>
<td><strong>Ocean</strong></td>
<td>$\partial_\tau U - c_1 YV + c_1 \partial_x H = c_1 \tau_x$</td>
</tr>
<tr>
<td></td>
<td>$YU + \partial_Y H = 0$</td>
</tr>
<tr>
<td></td>
<td>$\partial_\tau H + c_1(\partial_x U + \partial_Y V) = 0$</td>
</tr>
<tr>
<td><strong>SST</strong></td>
<td>$\partial_\tau T = -c_1 \zeta E_q + c_1 \eta H$</td>
</tr>
</tbody>
</table>
Atmosphere
\[ \partial_x K_A = -\chi_A E_q \left( 2 - 2 \bar{Q} \right)^{-1} \]
\[ - \partial_x R_A / 3 = -\chi_A E_q \left( 3 - 3 \bar{Q} \right)^{-1} \]
(B.C.) \[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

Ocean
\[ \partial_T K_O + c_1 \partial_x K_O = \chi_O c_1 \gamma (K_A - R_A) / 2 \]
\[ \partial_T R_O - (c_1 / 3) \partial_x R_O = -\chi_O c_1 \gamma (K_A - R_A) / 3 \]
(B.C.) \[ K_O(0, \tau) = r_W R_O(0, \tau) \]
\[ R_O(L_O, \tau) = r_E K_O(L_O, \tau) \]

SST
\[ \partial_T T = - c_1 \zeta E_q + c_1 \eta (K_O + R_O) \]
Solvability condition of the atmosphere model.

\[ \partial_x K_A = -\chi_A E_q (2 - 2\bar{Q})^{-1} \]
\[ - \partial_x R_A / 3 = -\chi_A E_q (3 - 3\bar{Q})^{-1} \]

\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

Intuition: With periodic boundary condition,

\[ \frac{1}{L_A} \int_0^{L_A} \partial_x K_A dx = K_A(L_A) - K_A(0) = 0, \]
\[ \frac{1}{L_A} \int_0^{L_A} \partial_x R_A dx = R_A(L_A) - R_A(0) = 0, \]

which implies

\[ \frac{1}{L_A} \int_0^{L_A} E_q dx = 0. \]
Solvability condition of the atmosphere model.

\[ + \partial_x K_A = -\chi_A (E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1} \]
\[ - \partial_x R_A/3 = -\chi_A (E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1} \]
\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

The absence of dissipation in the atmosphere imposes a solvability condition of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

\[ \frac{1}{L_A} \int_0^{L_A} E_q \, dx = 0. \]

In particular,

\[ E_q - \langle E_q \rangle = \begin{cases} \alpha_q T - \langle E_q \rangle & \text{for } x \in [0, L_O], \\ -\langle E_q \rangle & \text{for } x \in [L_O, L_A]. \end{cases} \]
Solvability condition of the atmosphere model.

\[ + \partial_x K_A = -\chi_A (E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1} \]
\[ - \partial_x R_A / 3 = -\chi_A (E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1} \]

\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

The absence of dissipation in the atmosphere imposes a solvability condition of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

\[ \frac{1}{L_A} \int_0^{L_A} E_q dx = 0. \]

In particular,

\[ E_q - \langle E_q \rangle = \begin{cases} \alpha_q T - \langle E_q \rangle & \text{for } x \in [0, L_O], \\ -\langle E_q \rangle & \text{for } x \in [L_O, L_A]. \end{cases} \]

Note that if \( K_A \) is a solution, then \( K_A + \text{const} \) is also a solution.
Solvability condition of the atmosphere model.

\[ d_A K_A + \partial_x K_A = -\chi_A (E_q - \langle E_q \rangle)(2 - 2\bar{Q})^{-1} \]
\[ d_A R_A - \partial_x R_A/3 = -\chi_A (E_q - \langle E_q \rangle)(3 - 3\bar{Q})^{-1} \]
\[ K_A(0, \tau) = K_A(L_A, \tau) \]
\[ R_A(0, \tau) = R_A(L_A, \tau) \]

The absence of dissipation in the atmosphere imposes a solvability condition of a zero equatorial zonal mean of latent heating forcing (Majda and Klein, 2003; Stechmann and Ogrosky, 2014). This reads:

\[ \frac{1}{L_A} \int_0^{L_A} E_q dx = 0. \]

In particular,

\[ E_q - \langle E_q \rangle = \begin{cases} 
\alpha_q T - \langle E_q \rangle & \text{for } x \in [0, L_O], \\
-\langle E_q \rangle & \text{for } x \in [L_O, L_A]. 
\end{cases} \]

Note that if \( K_A \) is a solution, then \( K_A + const \) is also a solution.

To guarantee the uniqueness of the solution, we add a tiny damping \( d_A \) into the model. With this tiny damping, the system has a unique solution with \( \langle K_A \rangle = \langle R_A \rangle = 0 \).

For numerical values, \( d_A = 10^{-8} \) while other coefficients are of order \( O(1) \).
Solving the coupled system using method of lines.

\[
\begin{align*}
\partial_\tau K_O + c_1 \partial_x K_O &= \frac{a}{2} (K_A - R_A), \quad K_O(0, \tau) = r_W R_O(0, \tau), \\
\partial_\tau R_O - \frac{c_1}{3} \partial_x R_O &= -\frac{a}{3} (K_A - R_A), \quad R_O(L_O, \tau) = r_E K_O(L_O, \tau), \\
\partial_\tau T &= -b T + c_1 \eta (K_O + R_O), \\
\frac{d}{d\tau} K_A + \partial_x K_A &= \frac{m_1}{\alpha_q} (1_{[0,L_O]} \alpha_q T - \langle E_q \rangle), \quad K_A(0, \tau) = K_A(L_A, \tau), \\
\frac{d}{d\tau} R_A - \frac{1}{3} \partial_x R_A &= \frac{m_2}{\alpha_q} (1_{[0,L_O]} \alpha_q T - \langle E_q \rangle), \quad R_A(0, \tau) = R_A(L_A, \tau),
\end{align*}
\]

where for notation simplicity we have defined
\[
\begin{align*}
a &= \chi_O c_1 \gamma, \quad b = c_1 \zeta \alpha_q, \quad m_1 = -\chi_A \alpha_q / (2 - 2\bar{Q}), \quad m_2 = -\chi_A \alpha_q / (3 - 3\bar{Q}),
\end{align*}
\]

Domain: \( K_O, R_O \) and \( T: [0, L_O] \), \( K_A \) and \( R_A: [0, L_A] \).

Define a vector \( u \) that contains all the prognostic variables at discrete equal-partitioned grid points,
\[
\begin{align*}
u &= (K_O; \ R_O; T) = (K_{O,1}, \ldots K_{O,N_O}, \ R_{O,1}, \ldots R_{O,N_O}, \ T_1, \ldots T_{N_O})^*.
\end{align*}
\]

Therefore, we solve the following equation
\[
\frac{du}{d\tau} = Mu.
\]

Recall: \( E_q = \alpha_q T \)
\[ \frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} \]

Let's start with the ocean model

\[ \partial_\tau K_O + c_1 \partial_x K_O = \frac{a}{2} (K_A - R_A), \quad K_O(0, \tau) = r_W R_O(0, \tau), \]

\[ \partial_\tau R_O - \frac{c_1}{3} \partial_x R_O = -\frac{a}{3} (K_A - R_A), \quad R_O(L_O, \tau) = r_E K_O(L_O, \tau), \]

An upwind scheme is utilized for the spatial discretization,

\[ \partial_\tau K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]

\[ \partial_\tau R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]

where the boundary conditions are given by \( K_{O,0} = r_W R_{O,1} \) and \( R_{O,N_O+1} = r_E K_{O,N_O} \).

**Upwind scheme.**

\[
\frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \quad \implies \quad \begin{cases} 
\frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & \text{for } a > 0 \\
\frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 & \text{for } a < 0
\end{cases}
\]

\[ u = u_0(a\tau - x) \implies \tau = \frac{x}{a} + \text{const.} \quad \text{Stability: } \left| \frac{a\Delta \tau}{\Delta x} \right| \leq 1. \]
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

Let’s start with the ocean model

\[ \partial_{\tau} K_O + c_1 \partial_x K_O = \frac{a}{2} (K_A - R_A), \]
\[ K_O(0, \tau) = r_W R_O(0, \tau), \]

\[ \partial_{\tau} R_O - \frac{c_1}{3} \partial_x R_O = -\frac{a}{3} (K_A - R_A), \]
\[ R_O(L_O, \tau) = r_E K_O(L_O, \tau), \]

An upwind scheme is utilized for the spatial discretization,

\[ \partial_{\tau} K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]
\[ \partial_{\tau} R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}), \quad i = 1, \ldots, N_O, \]

where the boundary conditions are given by \( K_{O,0} = r_W R_{O,1} \) and \( R_{O,N_O+1} = r_E K_{O,N_O} \).

**Upwind scheme.**

\[
\frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \begin{cases} \frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & \text{for } a > 0 \\ \frac{u_{i+1}^n - u_i^n}{\Delta \tau} + a \frac{u_i^n - u_{i+1}^n}{\Delta x} = 0 & \text{for } a < 0 \end{cases}
\]

\[ u = u_0(a\tau - x) \quad \Rightarrow \quad \tau = \frac{x}{a} + \text{const.} \]

Stability: \( |\frac{a\Delta \tau}{\Delta x}| \leq 1 \).
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

Let's start with the ocean model

\[
\begin{align*}
\partial_\tau K_O + c_1 \partial_x K_O &= \frac{a}{2} (K_A - R_A), \\
\partial_\tau R_O - \frac{c_1}{3} \partial_x R_O &= -\frac{a}{3} (K_A - R_A),
\end{align*}
\]

\[
K_O(0, \tau) = r_WR_O(0, \tau), \\
R_O(L_O, \tau) = r_EK_O(L_O, \tau),
\]

An upwind scheme is utilized for the spatial discretization,

\[
\begin{align*}
\partial_\tau K_O,i &= -\frac{c_1}{\Delta x} (K_O,i - K_O,i-1) + \frac{a}{2} (K_A,i - R_A,i), \\
\partial_\tau R_O,i &= \frac{c_1}{3\Delta x} (R_O,i+1 - R_O,i) - \frac{a}{3} (K_A,i - R_A,i),
\end{align*}
\]

where the boundary conditions are given by \( K_O,0 = r_WR_O,1 \) and \( R_O,N_O+1 = r_EK_O,N_O \).

**Upwind scheme.**

\[
\frac{\partial u}{\partial \tau} + a \frac{\partial u}{\partial x} = 0 \implies \begin{cases} 
\frac{u^{n+1} - u^n}{\Delta \tau} + a \frac{u^n - u^{n-1}}{\Delta x} = 0 & \text{for } a > 0 \\
\frac{u^{n+1} - u^n}{\Delta \tau} + a \frac{u^{n+1} - u^n}{\Delta x} = 0 & \text{for } a < 0
\end{cases}
\]

\[
u = u_0(a\tau - x) \implies \tau = \frac{x}{a} + \text{const.} \quad \text{Stability: } \left| \frac{a\Delta \tau}{\Delta x} \right| \leq 1.
\]
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[
\partial_\tau K_{O,i} = -\frac{c_1}{\Delta x} (K_{O,i} - K_{O,i-1}) + \frac{a}{2} (K_{A,i} - R_{A,i}) ,
\]

\[
\partial_\tau R_{O,i} = \frac{c_1}{3\Delta x} (R_{O,i+1} - R_{O,i}) - \frac{a}{3} (K_{A,i} - R_{A,i}) ,
\]

for \( i = 1, \ldots, N_O \). The boundary conditions are given by

\[
K_{O,0} = r_W R_{O,1}, \quad \text{and} \quad R_{O,N_O+1} = r_E K_{O,N_O}.
\]

\[
M_{11} = -\frac{c_1}{\Delta x} \begin{pmatrix} 1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ 1 \end{pmatrix}_{N_O \times N_O}, \quad M_{12} = -\frac{c_1}{\Delta x} \begin{pmatrix} -r_W \\ \vdots \\ 1 \\ -1 \\ 1 \end{pmatrix}_{N_O \times N_O},
\]

\[
M_{22} = \frac{c_1}{3\Delta x} \begin{pmatrix} 1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ 1 \end{pmatrix}_{N_O \times N_O}, \quad M_{23} = \frac{c_1}{3\Delta x} \begin{pmatrix} r_E \\ \vdots \\ 1 \\ -1 \\ -1 \end{pmatrix}_{N_O \times N_O},
\]

\[
M_{31} = \frac{c_1}{3\Delta x} \begin{pmatrix} 1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ 1 \end{pmatrix}_{N_O \times N_O}, \quad M_{33} = \frac{c_1}{3\Delta x} \begin{pmatrix} r_E \\ \vdots \\ 1 \\ -1 \\ -1 \end{pmatrix}_{N_O \times N_O}.
\]
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

Next, for the atmosphere model

\[
d_A K_A + \partial_x K_A = m_1 (1_{[0,L_O]} \alpha_q T - \langle E_q \rangle) / \alpha_q, \quad K_A(0, \tau) = K_A(L_A, \tau),
\]

\[
d_A R_A - \frac{\partial_x R_A}{3} = m_2 (1_{[0,L_O]} \alpha_q T - \langle E_q \rangle) / \alpha_q, \quad R_A(0, \tau) = R_A(L_A, \tau),
\]

The discretization results in

\[
d_A K_{A,i} + \frac{1}{\Delta x} (K_{A,i+1} - K_{A,i}) = m_1 \left( T_i - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right),
\]

\[
d_A R_{A,i} - \frac{1}{3\Delta x} (R_{A,i+1} - R_{A,i}) = m_2 \left( T_i - \frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right), \quad i = 1, \ldots, N_O,
\]

and

\[
d_A K_{A,i} + \frac{1}{\Delta x} (K_{A,i+1} - K_{A,i}) = m_1 \left( -\frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right),
\]

\[
d_A R_{A,i} - \frac{1}{3\Delta x} (R_{A,i+1} - R_{A,i}) = m_2 \left( -\frac{1}{N_A} \sum_{j=1}^{N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,
\]

where we have used the fact \( E_q = \alpha_q T \). Note that the averaging is taken for the whole equator and therefore the factor \( 1/N_A \) in front of the summation appears.
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

Rearranging the terms gives

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O}^{j \neq i} T_j \right),
\]

\[
-R_{A,i+1} + (3d_A \Delta x + 1)R_{A,i} = 3 \frac{N_A - 1}{N_A} \Delta x m_2 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O}^{j \neq i} T_j \right), \quad i = 1, \ldots N_O
\]

and

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right),
\]

\[
-R_{A,i+1} + (3d_A \Delta x + 1)R_{A,i} = 3 \frac{N_A - 1}{N_A} \Delta x m_2 \left( - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A
\]

Thus, \( K_{A,i} \) and \( R_{A,i} \) for \( i = 1, \ldots N_O \) can be solved via the following linear systems,

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T.
\]
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[
K_{A,i+1} + (d_A\Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = 1, \ldots, N_O,
\]

\[
K_{A,i+1} + (d_A\Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( -\frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,
\]

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T,
\]

where

\[
\beta = d_A\Delta x, \quad \tilde{C}_1 = \frac{N_A - 1}{N_A} \Delta x m_1, \quad \tilde{C}_2 = 3 \frac{N_A - 1}{N_A} \Delta x m_2,
\]

\[
M_K = \begin{pmatrix}
-1 + \beta & 1 & 0 & \cdots & 0 \\
-1 + \beta & -1 + \beta & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 + \beta & 1 \\
0 & 0 & \cdots & 0 & -1 + \beta \\
1 & 0 & \cdots & 0 & -1 + \beta
\end{pmatrix}_{N_A \times N_A}
\]

\[
K_A = \begin{pmatrix}
K_{A,1} \\
\vdots \\
K_{A,N_O} \\
\vdots \\
K_{A,N_A-1} \\
K_{A,N_A}
\end{pmatrix}_{N_A \times 1}
\]
\[
\frac{d}{d\tau}\begin{pmatrix} K_{O} \\ R_{O} \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_{O} \\ R_{O} \\ T \end{pmatrix}
\]

\[-R_{A,i+1} + (3d_{A}\Delta x + 1)R_{A,i} = 3\frac{N_{A} - 1}{N_{A}}\Delta x m_{2}\left(T_{i} - \frac{1}{N_{A} - 1}\sum_{1\leq j\leq N_{O}}T_{j}\right), \quad i = 1, \ldots, N_{O}\]

\[-R_{A,i+1} + (3d_{A}\Delta x + 1)R_{A,i} = 3\frac{N_{A} - 1}{N_{A}}\Delta x m_{2}\left(-\frac{1}{N_{A} - 1}\sum_{1\leq j\leq N_{O}}T_{j}\right), \quad i = N_{O} + 1, \ldots, N_{A}\]

\[
M_{K} \cdot K_{A} = \tilde{C}_{1}B_{K}T,
\]

\[
M_{R} \cdot K_{R} = \tilde{C}_{2}B_{R}T,
\]

where

\[
\beta = d_{A}\Delta x, \quad \tilde{C}_{1} = \frac{N_{A} - 1}{N_{A}}\Delta x m_{1}, \quad \tilde{C}_{2} = 3\frac{N_{A} - 1}{N_{A}}\Delta x m_{2},
\]

\[
M_{R} = \begin{pmatrix}
1 + 3\beta & -1 & & & \\
& 1 + 3\beta & -1 & & \\
& & & & \\
& & & & \\
-1 & & & & \\
\end{pmatrix}_{N_{A} \times N_{A}}, \quad R_{A} = \begin{pmatrix}
R_{A,1} \\
R_{A,2} \\
\vdots \\
R_{A,N_{O}} \\
R_{A,N_{O}-1} \\
R_{A,N_{A}} \\
\end{pmatrix}_{N_{A} \times 1}
\]
\[
\frac{d}{dT} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( T_i - \frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = 1, \ldots, N_O,
\]

\[
K_{A,i+1} + (d_A \Delta x - 1)K_{A,i} = \frac{N_A - 1}{N_A} \Delta x m_1 \left( -\frac{1}{N_A - 1} \sum_{1 \leq j \leq N_O} T_j \right), \quad i = N_O + 1, \ldots, N_A,
\]

\[M_K \cdot K_A = \tilde{C}_1 B_K T,\]

\[M_R \cdot K_R = \tilde{C}_2 B_R T,\]

where

\[
B_K = B_R = \begin{pmatrix} \frac{1}{N_A - 1} & \cdots & \frac{1}{N_A - 1} \\ \frac{1}{N_A - 1} & \cdots & \frac{1}{N_A - 1} \\ \vdots & \ddots & \vdots \\ \frac{1}{N_A - 1} & \cdots & \frac{1}{N_A - 1} \end{pmatrix}
\]

\[T = \begin{pmatrix} T_1 \\ \vdots \\ T_{N_O} \end{pmatrix}_{N_O \times 1}\]

Clearly, \(K_A\) and \(R_A\) can be expressed by \(T\).
\[
\frac{d}{d\tau} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} K_O \\ R_O \\ T \end{pmatrix}
\]

\[
M_K \cdot K_A = \tilde{C}_1 B_K T,
\]

\[
M_R \cdot K_R = \tilde{C}_2 B_R T.
\]

Recall the ocean model,

\[
\partial_\tau K_{O,i} = -\frac{c_1}{\Delta x} \left( K_{O,i} - K_{O,i-1} \right) + \frac{a}{2} \left( K_{A,i} - R_{A,i} \right),
\]

\[
\partial_\tau R_{O,i} = \frac{c_1}{3\Delta x} \left( R_{O,i+1} - R_{O,i} \right) - \frac{a}{3} \left( K_{A,i} - R_{A,i} \right),
\]

Since \( K_A \) and \( R_A \) can be expressed by \( T \), we have

\[
M_{13} = \frac{a}{2} \left[ \tilde{C}_1 M_K^{-1} B_K - \tilde{C}_2 M_R^{-1} B_R \right] \bigg|_{\text{Row } \{1:N_O\}},
\]

\[
M_{23} = -\frac{a}{3} \left[ \tilde{C}_1 M_K^{-1} B_K - \tilde{C}_2 M_R^{-1} B_R \right] \bigg|_{\text{Row } \{1:N_O\}},
\]

where the original matrices on the right hand side should be of size \( N_A \times N_O \) but only the first \( N_O \) rows are utilized to form \( M_{13} \) and \( M_{23} \) which correspond to \( K_{A,1}, \ldots, K_{A,N_O} \) and \( R_{A,1}, \ldots, R_{A,N_O} \) within the Pacific band.
\[
\frac{d}{d\tau} \begin{pmatrix}
K_0 \\
R_0 \\
T
\end{pmatrix} = \begin{pmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{23} & M_{33}
\end{pmatrix} \begin{pmatrix}
K_0 \\
R_0 \\
T
\end{pmatrix}
\]

Finally, let's consider the SST model,

\[
\partial_\tau T = -bT + c_1 \eta (K_0 + R_O),
\]

the discrete form of which is straightforward,

\[
\partial_\tau T_i = -bT_i + c_1 \eta (K_{O,i} + R_{O,i}).
\]

The matrices $M_{31}$, $M_{32}$ and $M_{33}$ are all diagonal and their $(i, i)$-th diagonal entries are given by

\[
(M_{31})_{ii} = (M_{32})_{ii} = c_1 \eta (x_i), \quad (M_{33})_{ii} = -b.
\]
Linear stability analysis

Recall that

\[ u = (K_0; R_0; T) = (K_{O,1}, \ldots K_{O,N_O}, R_{O,1}, \ldots R_{O,N_O}, T_1, \ldots T_{N_O})^*, \]

and

\[ \frac{du}{d\tau} = Mu. \]

Applying the eigen-decomposition of \( u \) gives

\[ \frac{du}{d\tau} = L\Lambda L^{-1}u, \]

where \( \Lambda \) is a block diagonal matrix (allowing Jordan blocks) with diagonal components being \( \lambda_i, i = 1, \ldots, 3N_O \). Define \( v = L^{-1}u \). Then we have

\[ \frac{dv}{d\tau} = \Lambda v, \quad \text{or componentwise for rank 1 Jordan blocks} \quad \frac{dv_i}{d\tau} = \lambda_i v_i. \]

Once \( v \) is solved, \( u \) can be easily recovered by

\[ u = \sum_{i=1}^{3N_O} L_i v_i, \]

where \( L_i \) is the \( i \)-th column of \( L \).

**Property:** If there is a \( v_j \) such that \( \| L_j v_j \| \gg \| L_i v_i \| \) for all \( i \neq j \), then

\[ u \approx L_j v_j. \]
The leading two modes are the ENSO modes.

All the eigenvalues $\lambda_j$ have negative real part $r_j < 0$. The coupled system is stable.

With $N_A = 64$ and $N_O = 28$, the frequency of the two ENSO modes is $0.22 \text{ year}^{-1}$ (4.5 years) and decay rate is $0.55 \text{ year}^{-1}$ (1.8 years).

The frequency and decay rate of the two ENSO modes are robust with respect to the number of $N_A$ and $N_O$. 

Linear solution:

$$
\frac{d\mathbf{u}}{d\tau} = M\mathbf{u} \rightarrow \frac{d\mathbf{v}}{d\tau} = \Lambda\mathbf{v}
$$

$$
\rightarrow \mathbf{v}_j = \mathbf{v}_j(0)e^{\lambda_j\tau} \quad \lambda_j = r_j + i\omega_j
$$
Robustness results with respect to the number of $N_A$ and $N_O$. 

(a) $N_A = 64$ 
(b) $N_A = 128$ 
(c) $N_A = 256$ 
(d) $N_A = 512$
Atmosphere:
\[- y v - \partial_x \theta = 0 \]
\[- y u - \partial_y \theta = 0 \]
\[- (\partial_x u + \partial_y v) = \frac{E_q}{1 - \overline{Q}} \]

Ocean:
\[ \partial_T U - c_1 Y V + c_1 \partial_x H = c_1 \tau_x \]
\[ Y U + \partial_Y H = 0 \]
\[ \partial_T H + c_1 (\partial_x U + \partial_Y V) = 0 \]

SST:
\[ \partial_T T = - c_1 \zeta E_q + c_1 \eta H \]

Linear solution with $N_A = 64$ and $N_O = 28$, where the decay rate is set to be zero for illustration purpose.
Summary of the model and model solutions:

- deterministic, linear and stable
- success of triggering regular ENSO cycles
- realistic interactions between atmosphere, ocean and SST

What lacks in the model solution ...

- stochasticity and irregularity
Stochastic Wind Bursts Parameterization

- Random atmospheric disturbances in the tropics, including westerly wind bursts, easterly wind bursts and the convective envelope of the MJO, are possible triggers to ENSO variability (Vecchi and Harrison, 2000; Tziperman and Yu, 2007; Hendon et al., 2007; Hu and Fedorov, 2015; Puy et al., 2016).

- All those atmospheric disturbances are usually more prominent in the equatorial Pacific prior to El Niño events.

(westerly \(\rightarrow\): from west to east \& easterly \(\leftarrow\): from east to west)
**Stochastic Wind Bursts:** in western Pacific depending on warm pool SST

Total wind stress \( \tau_x = \gamma (u + u_p) \), \( u_p \) : wind bursts,

Wind burst \( u_p = a_p(\tau) s_p(x) \phi_0(y) \) \( s_p \) : spatial structure

Evolution \( da_p/d\tau = -d_p a_p + \sigma_p(T_W) \dot{W}(\tau) \), \( T_W \) : western Pacific SST

**Markov Jump Process:** stochastic dependency on warm pool SST

Markov States \( \sigma_p(T_W) = \begin{cases} \sigma_{p0} & \text{quiescent} \\ \sigma_{p1} & \text{active} \end{cases} \)

States Switch

\( P(\text{quiescent} \rightarrow \text{active at } t + \Delta t) = r_{01} \Delta t + o(\Delta t) \)

\( P(\text{active} \rightarrow \text{quiescent at } t + \Delta t) = r_{10} \Delta t + o(\Delta t) \)

– State-dependent wind bursts, both westerly and easterly, depending on \( T_W \)
– No ad hoc prescription of wind burst thresholds.
Model Simulations: PDF and Power Spectrum

Moderate and Extreme El Niño events in the eastern Pacific, frequency $\approx 2$-$7$ years.

Model (5000 years)

PDF of Nino 3

- Truth
- Gaussian fit

Observations (1982–2016)

PDF of Nino 3

Spectrum of $T_E$

[Map of Nino regions with URL: https://www.ncdc.noaa.gov/]
The coupled model captures:

- quasi-regular moderate El Niño
- super El Niño mimicking 1997-1998 event
- super El Niño mimicking 2014-2016 event
Model Simulations: Hovmollers x-t

The coupled model captures:

▶ quasi-regular moderate El Niño
▶ super El Niño mimicking 1997-1998 event
▶ super El Niño mimicking 2014-2016 event
Mechanism of 1997-1998 El Niño

Observations 1997-1998

Model Simulation Mimicking 1997-1998

westerly (→)
easterly (←)
Mechanism of 2014-2016 El Niño

**Observations 2014-2016**

**Model Simulation Mimicking 2014-2016**

westerly (→)
easterly (←)
Model Simulations: Prediction Tests with Identical Initial Conditions

A

Zonal Winds  |  Thermocline Depth  |  SST

Year

150 200 250 m/s

150 200 250 m

150 200 250 K

Te, Tw

Wind Bursts

States

B

Zonal Winds  |  Thermocline Depth  |  SST

Year

150 200 250 m/s

150 200 250 m

150 200 250 K

Te, Tw

Wind Bursts

States

C

Zonal Winds  |  Thermocline Depth  |  SST

Year

150 200 250 m/s

150 200 250 m

150 200 250 K

Te, Tw

Wind Bursts

States

T_W  T_E
Starting from favorable recharged conditions,

- around 75% of events develop immediately, while
- around 20% of events may be delayed to the following year.

Delayed super El Niño events, at least in the present model, are statistically significant in the tropical Pacific and could reoccur in the future.