Practice problems for Midterm 2

Part 1 (pp. 61–75).

(1) Suppose the function $z(x, y)$ is given as the solution to equation $z^4+x^2+y^2 = 1$ defined around the point $x = 0, y = 0, z = 1$. Find $\nabla z(0, 0)$.

Diffrentiate identity $z^4(x, y) + x^2 + y^2 = 1$ in x and y. This gives

$$4z^3 z_x + 2x = 0,\ 4z^3 z_y + 2y = 0\,.$$

Substitute values $x = y = 0, z = 1$ and solve equations to get $z_x(0, 0) = 0, z_y(0, 0) = 0$. Thus, $\nabla z(0, 0) = (0, 0)$.

(2) Assume that $f(x, y, z) = x^2+y^3+z^3$ and let Σ be the level set for $f = 3$. Find, if possible, the tangent plane to this set at point $x = y = z = 1$. Explain.

Notice that $f(1, 1, 1) = 3$ and $\nabla f(1, 1, 1) = (2, 3, 3)$. Since the gradient is perpendicular to the level surface, $(2, 3, 3)$ is a normal vector to the tangent plane and equation for tangent plane is $2(x - 1) + 3(y - 1) + 3(z - 1) = 0$.

(3) For smooth function $f(x, y)$, the level set for $f = 1$ is the unit circle $x^2+y^2 = 1$. Can it happen that $\nabla f(1, 0) = (1, 2)$ or $\nabla f(1, 0) = (2, 0)$? Explain.

The gradient ∇f is perpendicular to the level curve so $\nabla f(1, 0)$ can not be equal to $(1, 2)$ since $(1, 2)$ is not perpendicular to the circle at point $(1, 0)$. Second case can happen, e.g., take $f = x^2+y^2$: $\nabla f = (2x, 2y)$ and $\nabla f(1, 0) = (2, 0)$.

Part 2 (pp. 83–98).

(1) For function $f(x, y, z) = \sin(xy^2e^z)$, find f_x, f_{xy}, f_{zz}.

(2) Find all critical points for the function $f(x, y) = (x + y)/(x^2 + y^2 + 1)$. Find $\max_{\mathbb{R}^2} f$ and $\min_{\mathbb{R}^2} f$. Explain.

System of equations for critical points is

$$x^2 + y^2 + 1 - 2x(x + y) = 0,\ x^2 + y^2 + 1 - 2y(x + y) = 0$$

Subtracting one equation from the other gives $(x - y)(x + y) = 0$. First case: $x = y$ gives two solutions $x = y = \pm 1/\sqrt{2}$. The second case, $x = -y$ gives no solutions. Thus, all critical polins are $(-1/\sqrt{2}, -1/\sqrt{2})$ and $(1/\sqrt{2}, 1/\sqrt{2})$.

We know that any point of max or min is one of critical points or it lies on the boundary. $f(-1/\sqrt{2}, -1\sqrt{2}) = -\sqrt{2}/2$ and $f(1/\sqrt{2}, 1\sqrt{2}) = \sqrt{2}/2$. Since $\lim_{(x,y)\to\infty} f(x, y) = 0$ (the proof of that was given in class), we have

$$\max f = \max\{0, -\sqrt{2}/2, \sqrt{2}/2\} = \sqrt{2}/2,\ \min f = \min\{0, -\sqrt{2}/2, \sqrt{2}/2\} = -\sqrt{2}/2\,.$$

(3) Find all extremal points of $f(x, y) = (x - 1)^6 + (y - 2)^6$.

If one does change of variable $s = (x - 1)^3, t = (y - 2)^3$, the function

$$\widetilde{f}(s, t) = s^2 + t^2$$

and it has the only extremal point (minimum) at $s = t = 0$ (follows from definition or second derivative test). Going back to original coordinates, $x = 1, y = 2$ is the only point of extremum (minimum).

(4) Find all extremal points of $f(x, y) = x^2 + x^3 + y^2$.

Critical points can be found from the system of equations:

$$2x + 3x^2 = 0,\ 2y = 0$$

and we have two solutions $(0, 0)$ and $(-2/3, 0)$.

Next, apply the second derivative test. For that, write the matrix of second derivatives:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2+6x & 0 \\ 0 & 2 \end{pmatrix}$$

For point $(0, 0)$, we have $f_{xx} = 2 > 0, \det = 2 \times 2 - 0 \times 0 = 4 > 0$ thus this is local min.

For point $(-2/3, 0)$, we have $f_{xx} = -2, \det = -4 < 0$ thus this is a saddle point (no local extremum).

Answer: $(0, 0)$ is the point of local min.

Part 3 (pp. 107–119).

(1) Find the volume of the set defined by $\{0 < x < 1, x^2 < y < 2x^2, 0 < z < xy\}$.

$$\int_0^1 \left(\int_{x^2}^{2x^2} xy dy \right) dx = \int_0^1 \frac{3x^5}{2} dx = \frac{1}{4}.$$

(2) Find the volume of the set defined by $\{0 < \theta < \pi/2, \theta < r < 2\theta, 0 < z < r\}$, where r and θ are polar coordinates in OXY plane.

$$\int_0^{\pi/2} \left(\int_\theta^{2\theta} r^2 dr \right) d\theta = \int_0^{\pi/2} \frac{7\theta^3}{3} d\theta = \frac{7}{12} \left(\frac{\pi}{2}\right)^4$$

(3) Compute

$$\int_{0<y<1, -y<x<e^y} (x^2 - 10y) dx dy.$$

$$\int_0^1 \left(\int_{-y}^{e^y} (x^2 - 10y) dx \right) dy = \int_0^1 \left(\frac{e^{3y}}{3} - 10ye^y + \frac{y^3}{3} - 10y^2 \right) dy =$$
$$\frac{e^3 - 1}{9} + \frac{1}{12} - \frac{10}{3} - 10 \int_0^1 ye^y dy$$

For the last integral, we integrate by parts

$$\int_0^1 ye^y dy = ye^y|_0^1 - \int_0^1 e^y dy = 1$$

Answer: $\frac{e^3-1}{9} + \frac{1}{12} - \frac{10}{3} - 10$.

Part 4 (pp. 121–132).

(1) Compute the average of the function $f(x, y, z) = \sin x + \sin y + \sin z$ over the set $D = \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}$.

By symmetry, the integral is equal to

$$3\int_0^\pi \left(\int_0^\pi \left(\int_0^\pi \sin z dz\right) dy\right) dx = 6\pi^2 .$$

Average $= 6\pi^2/(\pi)^3$.

(2) Let D be the annulus given by $1 < x^2 + y^2 + z^2 < 4$. Find

$$\int_D |x| dxdydz .$$

By symmetry,

$$\int_D |x|dxdydz = \int_D |z|dxdydz = \int_1^2 \left(\int_0^{2\pi} d\theta \left(\int_0^\pi r^3 \sin\phi |\cos\phi| d\phi\right) d\theta\right) dr$$
$$= 2\pi \int_1^2 r^3 dr = \frac{15\pi}{2} .$$

(3) Problems 7,8,14,16 on page 134.

here is problem 16: equation $x^2 + y^2 - 2x = 0$ can be rewritten as $(x-1)^2 + y^2 = 1$ which is the circle of radius 1 centered at $(1, 0)$.

One way to calculate the integral: parameterize the point inside the circle as $x = 1 + r\cos\theta, y = r\sin\theta$, $\theta \in [0, 2\pi), 0 < r < 1$. Then, $z^2 = 4 - x^2 - y^2 = 3 - 2r\cos\theta - r^2$ and the volume can be computed in cylindrical coordinates as

$$\int_0^1 \left(\int_0^{2\pi} \left(\int_0^{\sqrt{3-2r\cos\theta-r^2}} zrdz\right) d\theta\right) dr$$
$$= \int_0^1 \frac{r}{2} \left(\int_0^{2\pi} (3 - 3r\cos\theta - r^2) d\theta\right) dr = \pi\int_0^1 r(3 - r^2) dr = \frac{5\pi}{4} .$$