Practice problems for Midterm 1

Part 1 (chapter 1).

(1) Find the volume of the parallelepiped spanned by the vectors AB, AC, AD where $A = (1,2,0), B = (2,3,1), C = (-1,0,3), D = (0,2,0)$.

$AB = (1,1,1), AC = (-2,-2,3), AD = (-1,0,0)$, then

$$\text{vol} = \left| \det \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & 3 \\ -1 & 0 & 0 \end{pmatrix} \right| = 5 .$$

(2) Write defining equation of the plane that contains the origin and the line given by $x = 3 - 2t, y = 1 + t, z = t$.

Take $t = 0$ to see that $A = (3,1,0)$ belongs to the plane. Since O also belongs to it, $OA = (3,1,0)$ is parallel to it along with $(-2,1,1)$ which parameterizes the line. Thus, the normal is

$$n = \det \begin{pmatrix} i & j & k \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} = i - 3j + 5k = (1,-3,5) .$$

and equation is $x - 3y + 5z = 0$ since it contains the origin.

(3) Write equation for the line which goes through the origin and is perpendicular to the plane that contains points $A = (1,2,0), B = (2,3,1), C = (-1,0,3)$.

$AB = (1,1,1), AC = (-2,-2,3)$ so the normal is given by

$$n = \det \begin{pmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & -2 & 3 \end{pmatrix} = 5i - 5j + 0k = (5,-5,0) .$$

Scale by $1/5$. The parametric equation for the line is $x = t, y = -t, z = 0$.

(4) Find the distance from point $A = (1,2,1)$ to the plane $x + y - z = 3$.

We know that the distance from the origin to the plane $ax + by + cz = d$ is given by $|d|/\sqrt{a^2 + b^2 + c^2}$. Change variables in the equation to place A to the origin in new coordinates: $x' = x - 1, y' = y - 2, z' = z - 1$. Equation takes the form $x' + y' - z' = 1$ and the distance is $1/\sqrt{3}$ as follows from the formula.

(5) (*) Write the equation of the line which is bisector of two lines: one contains A and B, the other one contains A and C. We have $A = (1,1,2), B = (2,1,0), C = (1,0,0)$.

$$AB = (1,0,-2), \frac{AB}{\|AB\|} = (\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}),$$

$$AC = (0, -1, -2), \frac{AC}{\|AC\|} = (0, -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$$

Bisector is a diagonal in rombus so we take

$$\frac{AB}{\|AB\|} + \frac{AC}{\|AC\|} = (1/\sqrt{5}, 0, -2/\sqrt{5}) + (0, -1/\sqrt{5}, -2\sqrt{5})$$
$$= (1/\sqrt{5}, -1/\sqrt{5}, -4\sqrt{5})$$

Scale by $\sqrt{5}$ to get $(1, -1, -4)$ and equation takes the form $x = 1 + t, y = 1 - t, z = 2 - 4t$.

The bisector of the other angle between the lines is easy to find too: it comes from

$$-\frac{AB}{\|AB\|} + \frac{AC}{\|AC\|} = (-1/\sqrt{5}, 0, 2/\sqrt{5}) + (0, -1/\sqrt{5}, -2\sqrt{5})$$
$$= (-1/\sqrt{5}, -1/\sqrt{5}, 0)$$

and the equation is $x = 1 - t, y = 1 - t, z = 2$.

Part 2 (chapter 2).

(1) For the curve given by equation $|x|^3 + |y|^3 = 2$, find the tangent line at point $(1, 1)$.

Parameterize by $x = 2^{1/3} \cos^{2/3} \phi, y = 2^{1/3} \sin^{2/3} \phi$. Point $(1, 1)$ corresponds to $\phi_0 = \pi/4$, tangent vectors at $(1, 1)$ is $v = 2/3(-1, 1)$ so the tagent line is $x + y = 2$.

(2) Compute the length of the curve $r = (1 + \cos\phi), \phi \in [0, 2\pi]$, given in polar coordinates. Draw the picture.

$$l = \int_0^{2\pi} \sqrt{\sin^2\phi + (1 + \cos\phi)^2} d\phi = 4 \int_0^{\pi} \cos(\phi/2) d\phi = 8\,.$$

(3) For the curve $(\sin t, \cos t, \cosh t)$, find the unit tangent, curvature vector and curvature at point $t = 0$.

See calculations in notes.

(4) The position of a point at time t is given by $(t, \sin t, \cos t)$. The motion started when $t = 0$. At what time the point will travel 1 unit?

$$l(t) = \int_0^t \sqrt{2} d\tau = \sqrt{2} t$$

Equation $l(t) = 1$ has solution $t = 1/\sqrt{2}$.

(5) Find the length of the curve $(x, x^{3/2}), x \in [0,1]$.

$$l = \int_0^1 \sqrt{1 + 9x/4}dx = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$$

Part 3 (chapter 3).

(1) The function $f = xy^2$. Find the tangent line to the following level set: $f = 1$ at point $(1,1)$.

Level set is $xy^2 = 1$ or a curve (y^{-2}, y). Tangent vector is $(-2,1)$ and tangent line is $(1-2t, 1+t)$.

(2) Classify the quadratic forms: $2x^2 + 30y^2 + 5xy, 2xy - 10y^2, x^2 - 2xy + y^2$.

See calculations in the lecture notes.

(3) Find the domain and the range of the function $f(x,y) = \sin(\sqrt{xy}) + (x-y)^3$. Explain your answer.

Domain: $xy \geq 0$. Range: notice that $f(x,0) = x^3$ and the range of this function is $\mathbb{R}$ so the range for f is $\mathbb{R}$ as well (it can not be larger than that).

Part 4 (chapter 4, pp. 49–61).

(1) Find equation for the tangent plane for the graph of $f(x,y) = \sin(xy) + x^2 + y^2$ at point $(1,1)$.

See calculations in the lecture notes.

(2) Find the directional derivative of $f(x,y) = e^{x^2+y^4}$ at point $(1,1)$ in the direction $v = (1,-1)$.

For the gradient,

$$\nabla f(1,1) = e^2(2,4)$$

so directional derivative is

$$\nabla f \cdot \frac{v}{\|v\|} = -\sqrt{2}e^2$$

(3) Find partial derivatives of the following functions $f = \log(x^2 + y^6 + 1)$, $f = \arctan(x^2 + y^2)$.

See calculations in the notes.

(4) (*) Function $f(r,\phi) = r\cos(\phi^2)$ in polar coordinates. Find $f_y(1,1)$.

$r = \sqrt{x^2+y^2}, \phi = \arctan(y/x),\ f(r,\phi) = \sqrt{x^2+y^2}\cos(\arctan^2(y/x))\,.$
Chain rule gives

$$f_y(1,1) = \frac{1}{\sqrt{2}}\cos((\pi/4)^2) - \frac{\pi}{2\sqrt{2}}\sin((\pi/4)^2)\,.$$