MINIMAL DEGREES OF FINITE SIMPLE GROUPS OF LIE TYPE

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For a finite simple group L of Lie type, d denotes the order of the group of "diagonal automorphisms," and $P(L)$ is the least degree of a faithful permutation representation of L, i.e. the least integer N such that L is isomorphic to a transitive subgroup of S_N.

Table 1a. Simple classical groups of Lie type (taken from [1]).

L	d	$\|L\|$	$P(L)$
$\mathrm{PSL}_n(q)^{\dagger\dagger}, n \geq 2$	$(n, q-1)$	$\frac{1}{d} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - 1)$	$\frac{q^n - 1}{q - 1}$ ‡
$\mathrm{PSp}_{2m}(q), m \geq 2, q \geq 3$	$(2, q-1)$	$\frac{1}{d} q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)$	$\frac{q^{2m} - 1}{q - 1}$ ‡
$\mathrm{Sp}_{2m}(2)^{\dagger\dagger}, m \geq 3$	1	$2^{m^2} \prod_{i=1}^{m} (2^{2i} - 1)$	$2^{m-1}(2^m - 1)$
$\mathrm{P}\Omega_{2m+1}(q), m \geq 3, q$ odd, ≥ 5	2	$\frac{1}{2} q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)$	$\frac{q^{2m} - 1}{q - 1}$
$\mathrm{P}\Omega_{2m+1}(3), m \geq 3$	2	$\frac{1}{2} 3^{m^2} \prod_{i=1}^{m} (3^{2i} - 1)$	$\frac{1}{2} 3^m (3^m - 1)$
$\mathrm{P}\Omega^+_{2m}(q), m \geq 4, q \geq 3$	$(4, q^m - 1)$	$\frac{1}{d} q^{m(m-1)} (q^m - 1) \prod_{i=1}^{m-1} (q^{2i} - 1)$	$\frac{(q^m - 1)(q^{m-1} + 1)}{q - 1}$
$\mathrm{P}\Omega^+_{2m}(2), m \geq 4$	$(4, q^m - 1)$	$\frac{1}{d} q^{m(m-1)} (q^m - 1) \prod_{i=1}^{m-1} (q^{2i} - 1)$	$\frac{(q^m - 1)(q^{m-1} + 1)}{q - 1}$
$\mathrm{P}\Omega^-_{2m}(q), m \geq 4$	$(4, q^m + 1)$	$\frac{1}{d} q^{m(m-1)} (q^m + 1) \prod_{i=1}^{m-1} (q^{2i} - 1)$	$\frac{(q^m + 1)(q^{m-1} - 1)}{q - 1}$
$\mathrm{PSU}_3(q)^{\dagger\dagger}$	$(3, q+1)$	$\frac{1}{d} q^6 \prod_{i=2}^{3} (q^i - (-1)^i)$	$q^3 + 1$
$\mathrm{PSU}_4(q)$	$(4, q+1)$	$\frac{1}{d} q^6 \prod_{i=2}^{4} (q^i - (-1)^i)$	$(q^3 + 1)(q + 1)$
$\mathrm{PSU}_n(q), n \geq 5, (n, q) \neq (6m, 2)$	$(n, q+1)$	$\frac{1}{d} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - (-1)^i)$	$\frac{(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})}{q^2 - 1}$ ‡
$\mathrm{PSU}_n(2), 6 \mid n$	$(n, q+1)$	$\frac{1}{d} q^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (q^i - (-1)^i)$	$\frac{2^{n-1}(2^n - 1)}{3}$

††N.B. The groups $\mathrm{PSL}_2(2)$, $\mathrm{PSL}_2(3)$, $\mathrm{PSU}_3(2)$, and $\mathrm{Sp}_4(2)$ are not simple.

‡See footnote on next page.

Table 1b. Simple exceptional groups of Lie type (taken from [2], [3], and [4]).

$G_2(q)$	1	$q^6(q^2-1)(q^6-1)$	$\frac{q^6-1}{q-1}$ ‡
$F_4(q)$	1	$q^{24}\prod_{i=2,6,8,12}(q^i-1)$	$\frac{(q^{12}-1)(q^4+1)}{q-1}$
$E_6(q)$	$(3,q-1)$	$\frac{1}{d}q^{36}\prod_{i=2,5,6,8,9,12}(q^i-1)$	$\frac{(q^9-1)(q^8+q^4+1)}{q-1}$
$E_7(q)$	$(2,q-1)$	$\frac{1}{d}q^{63}\prod_{i=2,6,8,10,12,14,18}(q^i-1)$	$\frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1}$
$E_8(q)$	1	$q^{120}\prod_{i=2,8,12,14,18,20,24,30}(q^i-1)$	$\frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)}{q-1}$
${}^2B_2(q), q=2^{2m+1}$	1	$q^2(q^2+1)(q-1)$	q^2+1
${}^2G_2(q), q=3^{2m+1}$	1	$q^3(q^3+1)(q-1)$	q^3+1
${}^2F_4(q), q=2^{2m+1}$	1	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$(q^6+1)(q^3+1)(q+1)$
${}^3D_4(q)$	1	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$(q^8+q^4+1)(q+1)$
${}^2E_6(q)$	$(3,q+1)$	$\frac{1}{d}q^{36}\prod_{i=2,5,6,8,9,12}(q^i-(-1)^i)$	$\frac{(q^{12}-1)(q^6-q^3+1)(q^4+1)}{q-1}$

Table 2. Values of $P(L)$ that deviate from the formulas above (taken from [1], [2], and [4]).

L	$\lvert L\rvert$	$P(L)$
$\mathrm{PSL}_2(5)$	60 ($\cong A_5$)	5
$\mathrm{PSL}_2(7)$	168	7
$\mathrm{PSL}_2(3^3)$	360 ($\cong A_6$)	6
$\mathrm{PSL}_2(11)$	660	11
$\mathrm{PSL}_4(2)$	$\frac{1}{2}8!$ ($\cong A_8$)	8
$\mathrm{PSp}_4(3)$	$2^6\cdot 3^4\cdot 5$	27
$\mathrm{Sp}_4(2)'$	360 ($\cong A_6$)	6
$\mathrm{PSU}_3(5)$	$2^4\cdot 3^2\cdot 5^3\cdot 7$	50
$G_2(3)$	$2^6\cdot 3^6\cdot 7\cdot 13$	351
$G_2(2^2)$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	416
${}^2F_4(2)'$	$2^{11}\cdot 3^3\cdot 5^2\cdot 11$	1600

‡There are 11 values of $P(L)$ which deviate from those given in Table 1. Correct values are given in Table 2, as well as for $\mathrm{Sp}_4(2)'$ and ${}^2F_4(2)'$ (the Tits group), the only finite simple groups of Lie type not appearing in Table 2.

References

[1] Peter Kleidman and Martin Liebeck. *The subgroup structure of the finite classical groups*, volume 129 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.
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[4] A. V. Vasil′ev. Minimal permutation representations of finite simple exceptional groups of twisted type. *Algebra i Logika*, 37(1):17–35, 122, 1998.

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