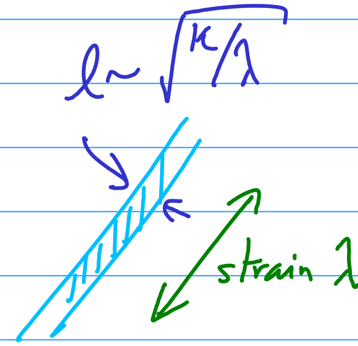
IMA Tutorial: Transport & Mixing

2010/02/17

Lecture 3: Effective Diffusivity



Recall: filaments in chaotic advection

Goal was to compute decay of variance,

$$\langle \theta^2 \rangle \sim e^{-\gamma t} \qquad (\gamma = \lambda \text{ for uniform strain})$$

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Longrightarrow \quad \frac{\partial \theta}{\partial t} = K_{\text{eff}} \nabla^2 \theta \; ?$$

Diffusion arises from noise: $\quad x_n = x_{n-1} + \xi_n$

Assume $\langle \xi_n \rangle = 0, \quad \langle \xi_n^2 \rangle = \sigma^2$ i.i.d. (Gaussian)

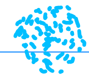
$$x_n = \underbrace{x_0}_{0} + \sum_{i=1}^{n} \xi_n, \qquad \langle x_n \rangle = 0$$

$$\langle x_n^2 \rangle = \sum_{i=1}^{n} \langle \xi_n^2 \rangle = \overset{\sim \text{"time"}}{n} \sigma^2 = \underbrace{2Kt}_{\text{by definition}}$$

In $\underline{\underline{d}}$ dimensions, $\quad t = nT$

$$\langle x_n^2 + y_n^2 \; (+ z_n^2) \rangle = n \underline{\underline{d}} \sigma^2 = 2dKnT$$

$$\boxed{K_{\text{eff}} = \frac{\sigma^2}{2T}}$$

Now if we take a "cloud" of points , and define a density

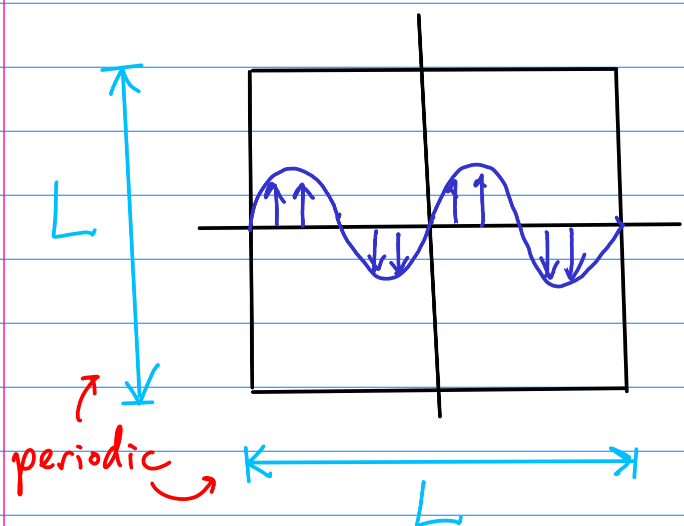
$$\theta(\underline{x}, t) = \text{density of points}$$

then θ satisfies $\dfrac{\partial \theta}{\partial t} = K \nabla^2 \theta$ if each point evolves independently according to $x' = x + \xi$.

Of course, this requires "coarse-graining": it is only true if we don't look to closely (scales $\lesssim \sigma$) or too often (time scales $\lesssim T$).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous SINE FLOW.



- Velocity field (shear flow)

$$\underline{u}_H = \left(U \sin\left(\frac{2\pi k y}{L}\right), 0 \right)$$ STEP 1

applied for $0 \lesssim t < \tau/2$

- $$\underline{u}_V = \left(0, U \sin\left(\frac{2\pi k x}{L}\right) \right)$$ STEP 2

for $\tau/2 \lesssim t < \tau$.

Can solve $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$ exactly:

STEP 1: $$x(\tau/2) = x_0 + U\tau/2 \sin\left(\frac{2\pi k y_0}{L}\right)$$

$$y(\tau/2) = y_0$$

STEP 2: $x(\tau) = x(\tau/2)$

$x(\tau/2) = x(\tau)$

$$y(\tau) = y(\tau/2) + \frac{U\tau}{2}\sin\left(\frac{2\pi k x(\tau/2)}{L}\right)$$

Write as one map of period τ:

$$\boxed{\begin{aligned} x' &= x + T\sin(2\pi k y/L) \\ y' &= y + T\sin(2\pi k x'/L) \end{aligned}}$$

$$T \equiv \frac{U\tau}{2}$$

Easy to iterate on a gazillion particles.

note x'! Important for area-preservation (comes from incompressibility)

Example 1: Run Matlab script example(1).

$$L = k = 1, \quad T = 0.1$$

Note how regular the orbits are: for small T the map is effectively a symplectic integrator

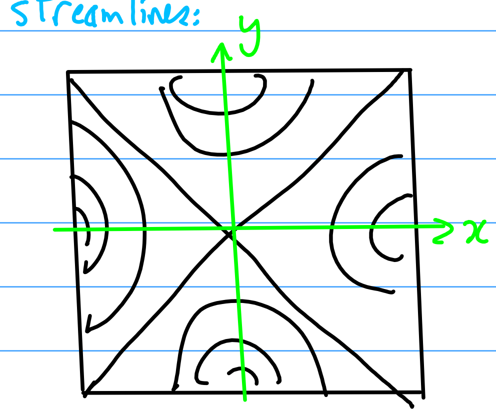
$$\frac{x'-x}{T} = \sin\left(\frac{2\pi k y}{L}\right), \quad \frac{y'-y}{T} = \sin\left(\frac{2\pi k x'}{L}\right)$$

As $T \to 0$, this approximates $\frac{dx}{dt} = \sin\left(\frac{2\pi k y}{L}\right)$, $\frac{dy}{dt} = \sin\left(\frac{2\pi k x}{L}\right)$,

$= \partial\psi/\partial y$ $\qquad = -\partial\psi/\partial x$

or flow with streamfunction: $\psi = \frac{L}{2\pi k}\left(\cos\left(\frac{2\pi k x}{L}\right) - \cos\left(\frac{2\pi k y}{L}\right)\right)$

streamlines:



The streamlines aren't traced exactly because T is finite.

<u>Example 2</u> adds a bit of noise.

$$x' = (\text{sine map}) + \sqrt{2D}\,\xi$$

Gaussian random var. with $\langle \xi^2 \rangle = 1$.

<u>Example 3:</u> $T = 1$. Now doesn't approximate a flow at all $\rightarrow$ CHAOTIC.

<u>Example 4:</u> $T = 1$, $L = 1$, $D = 10^{-4}$: "fat" filaments.

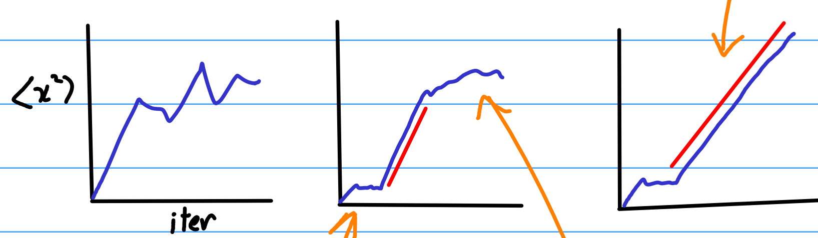
$\rightarrow$ measure width by clicking

$\rightarrow$ repeat for $D = 10^{-6}$

$\rightarrow$ observe rough $\sqrt{D}$ scaling for filament width (see Lecture 1)

<u>Example 5:</u> $T = 1/2$, $k = 1$, $D = 10^{-6}$, make L larger.

Plot $\langle x^2 \rangle$ vs iteration



$L = 1$ $\quad$ $L = 3$ $\quad$ $L = 25$

particles are initially very close, so correlated

"chaotic mixing" regime

particles reach sides of box

Hence, the concept of an effective diffusivity makes sense if we look at large scales, such that we cannot see the correlated small scale motions, and long times. (but not too long!)

$\rightarrow$ <u>Useful for turbulence</u>

$$K_{eff} \sim 0.068 \gg D = 10^{-6}$$

<u>Note that the "cross" shape  evident in the pattern is not captured.</u>