Eulerian and Lagrangian Pictures of Mixing

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Experiment of Rothstein et al.: Persistent Pattern

Disordered array of magnets with oscillatory current drive a thin layer of electrolytic solution.

periods $2, 20, 50, 50.5$

[Rothstein, Henry, and Gollub, Nature **401**, 770 (1999)]
Evolution of Pattern

- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)
Local vs Global Regimes of Mixing

Local theory:

• Based on distribution of Lyapunov exponents.
Local vs Global Regimes of Mixing

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- [Antonsen et al., Phys. Fluids (1996)] 
  Average over angles 
- [Balkovsky and Fouxon, PRE (1999)] 
  Statistical model 
- [Son, PRE (1999)] 
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Global theory:

• Eigenfunction of advection–diffusion operator.

• [Pierrehumbert, Chaos Sol. Frac. (1994)] Strange eigenmode
  [Fereday et al., Wonhas and Vassilicos, PRE (2002)] Baker’s map
  [Sukhatme and Pierrehumbert, PRE (2002)]
  [Fereday and Haynes (2003)] Unified description
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- Eigenfunction of advection–diffusion operator.
- So far, local theories are **Lagrangian** and global theories are **Eulerian**.
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• Today: Try to connect the two pictures.
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Global theory:

- Eigenfunction of advection–diffusion operator.
- So far, local theories are Lagrangian and global theories are Eulerian.
- Today: Try to connect the two pictures.
- Cannot often do this! Map allows (mostly) analytical results.
A Bit of History

Eulerian *(spatial)* coordinates are due to...
Eulerian (\textit{spatial}) coordinates are due to... 

d’Alembert
A Bit of History

... and Lagrangian (material) coordinates to...

d’Alembert
A Bit of History

... and Lagrangian (material) coordinates to...
The people responsible for the confusion...
The people responsible for the confusion... 

Lagrange

Dirichlet

(See footnote in Truesdell, *The Kinematics of Vorticity.* )
The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0, 1]^2$,

$$\mathcal{M}(x) = \mathbb{M} \cdot x + \phi(x),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(x) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot x$ is the **Arnold cat map**.

The map $\mathcal{M}$ is **area-preserving and chaotic**.

For $\varepsilon = 0$ the stretching of fluid elements is **homogeneous in space**.

For small $\varepsilon$ the system is still **uniformly hyperbolic**.
Advection and Diffusion: Eulerian Viewpoint

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution \( \theta^{(i-1)}(x) \),

\[
\theta^{(i)}(x) = \mathcal{H}_\kappa \theta^{(i-1)}(\mathcal{M}^{-1}(x))
\]

where \( \kappa \) is the diffusivity, with the heat operator \( \mathcal{H}_\kappa \) and kernel \( h_\kappa \),

\[
\mathcal{H}_\kappa \theta(x) := \int_{\mathbb{T}^2} h_\kappa(x - y) \theta(y) \, dy;
\]

\[
h_\kappa(x) = \sum_k \exp(2\pi ik \cdot x - k^2 \kappa).
\]

In other words: advect instantaneously and then diffuse for one unit of time.
Transfer Matrix

Fourier expand $\theta^{(i)}(x)$,

$$\theta^{(i)}(x) = \sum_k \hat{\theta}_k^{(i)} e^{2\pi i k \cdot x}.$$  

The effect of advection and diffusion becomes

$$\hat{\theta}_k^{(i)}(x) = \sum_q T_{kq} \hat{\theta}_q^{(i-1)},$$

with the transfer matrix,

$$T_{kq} := \int_{T^2} \exp \left( 2\pi i (q \cdot x - k \cdot M(x)) - \kappa q^2 \right) \, dx,$$

$$= e^{-\kappa q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} \left((k_1 + k_2) \varepsilon\right), \quad Q := k \cdot M - q,$$

where the $J_Q$ are the Bessel functions of the first kind.
In the absence of diffusion ($\kappa = 0$) the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} |\theta^{(i)}(\boldsymbol{x})|^2 \, d\boldsymbol{x} = \sum_k \sigma_k^{(i)}, \quad \sigma_k^{(i)} := |\hat{\theta}_k^{(i)}|^2$$

is preserved. (We assume the spatial mean of $\theta$ is zero.) For $\kappa > 0$ the variance decays.

We consider the case $\kappa \ll 1$, of greatest practical interest.
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Variance: A Measure of Mixing

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Three phases:

- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.
Decay of Variance

\[ \varepsilon = 10^{-3} \]

\[ e^{-15.2i} \]

\[ \kappa = 10 \]

\[ 10^{-5} \]

\[ 10^{-2} \]

\[ 0.5 \]

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Variance: 5 iterations for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$
Eigenfunction for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

(Renormalised by decay rate)
Decay Rate

For small $\varepsilon$, the dominant Bessel function is $J_1$, so the decay factor $\mu^2$ for the variance is given by

$$\mu = \left| T_{(0 \ 1), (0 \ 1)} \right| = e^{-\kappa} J_1 (\varepsilon) = \frac{1}{2} \varepsilon + \mathcal{O}(\kappa \varepsilon, \varepsilon^2).$$

Hence, for small $\varepsilon$ the decay rate is limited by the $(0 \ 1)$ mode. The decay rate is independent of $\kappa$ for $\kappa \to 0$. 
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This is an analogous result to the baker’s map [Fereday et al., Wonhas and Vassilicos, PRE (2002)]. Here the agreement with numerical results is good for $\varepsilon$ quite close to unity.
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In the baker’s map the discontinuity imply a slow convergence of the Fourier modes. However, it is a one-dimensional problem.
Decay Rate as $\kappa \to 0$
Lagrangian Viewpoint

- Puzzle: Superexponential decay in Lagrangian coordinates.
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Discover what large-scale eigenfunction looks like in Lagrangian coordinates (hint: they are not eigenfunctions!).
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Why do this? The two viewpoints are a priori unrelated, because they for these highly-chaotic systems they are connected by an extremely convoluted transformation!
Lagrangian Viewpoint

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- Discover what large-scale eigenfunction looks like in Lagrangian coordinates (hint: they are not eigenfunctions!).
- Why do this? The two viewpoints are a priori unrelated, because they for these highly-chaotic systems they are connected by an extremely convoluted transformation!
- But must give same answer for a scalar quantity like the decay rate.
Advection-diffusion (A–D) equation:

\[ \partial_t \theta + \mathbf{v} \cdot \partial_x \theta = \kappa \partial_x^2 \theta. \]
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We define Lagrangian coordinates \( \mathbf{X} \) by

\[
\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{X}.
\]
Advection-diffusion (A–D) equation:

\[ \partial_t \theta + \mathbf{v} \cdot \partial_x \theta = \tilde{\kappa} \partial_{xx}^2 \theta. \]

We define Lagrangian coordinates \( \mathbf{X} \) by

\[ \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{X}. \]

Transform A–D equation to Lagrangian coordinates,

\[ \dot{\theta} = \partial_{\mathbf{X}} (\mathbb{D} \cdot \partial_{\mathbf{X}} \theta). \]

Anisotropic diffusion tensor, in terms of metric or Cauchy–Green strain tensor:

\[ \mathbb{D} := \tilde{\kappa} \mathbf{g}^{-1}; \quad g_{pq} := \sum_i \frac{\partial x^i}{\partial X^p} \frac{\partial x^i}{\partial X^q}. \]
Velocity field doesn’t enter the Lagrangian equation directly: regard the time dependence in $\mathbb{D}$ as given by map rather than flow. The solution of the A–D equation in Fourier space is then

$$
\hat{\theta}^{(i)}_{k} = \sum_{\ell} \exp(G^{(i)})_{k\ell} \hat{\theta}^{(i-1)}_{\ell},
$$

where $i$ denotes the $i$th iterate of the map, and

$$
G^{(i)}_{k\ell} = -4\pi^2 T \int_{\mathbb{T}^2} (k \cdot \mathbb{D}^{(i)} \cdot \ell) e^{-2\pi i (k-\ell) \cdot X} d^2 X.
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Velocity field doesn’t enter the Lagrangian equation directly: regard the time dependence in $\mathcal{D}$ as given by map rather than flow. The solution of the A–D equation in Fourier space is then

$$\hat{\theta}^{(i)}_k = \sum_\ell \exp(\mathcal{G}^{(i)}_{k\ell}) \hat{\theta}^{(i-1)}_\ell,$$

where $i$ denotes the $i$th iterate of the map, and

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This is an exact result, but the great difficulty lies in calculating the exponential of $\mathcal{G}^{(i)}$. We shall accomplish this perturbatively.
\[ \mathcal{M}(x) = \mathbb{M} \cdot x + \phi(x), \]
\[ \mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(x) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix}; \]

The eigenvalues of \( \mathbb{M} \) are
\[ \Lambda_u = \Lambda = \frac{1}{2}(3 + \sqrt{5}) = \cot^2 \theta, \quad \Lambda_s = \Lambda^{-1} = \frac{1}{2}(3 - \sqrt{5}) = \tan^2 \theta \]
and the corresponding eigenvectors,
\[ (\hat{u} \ \hat{s}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]
\[ \mathcal{M}(\bm{x}) = \mathbb{M} \cdot \bm{x} + \phi(\bm{x}), \]

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\( \Lambda^{-1} \) Contract

\( \Lambda \) Stretch
The coefficients of expansion and characteristic directions for the linear cat map are uniform in space. Perturb off this.

To leading order in $\varepsilon$, the coefficient of expansion is written as

$$
\Lambda^{(i)}_\varepsilon = \Lambda^i (1 + \varepsilon \eta^{(i)})
$$

where $\Lambda$ is the coefficient of expansion for the unperturbed cat map; the perturbed eigenvectors are similarly written

$$
\hat{u}^{(i)}_\varepsilon = \hat{u} + \varepsilon \zeta^{(i)} \hat{s}, \quad \hat{s}^{(i)}_\varepsilon = \hat{s} - \varepsilon \zeta^{(i)} \hat{u}.
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Simple application of matrix perturbation theory to Jacobian matrix of the map. The symmetrised Jacobian is the metric:

$$g^{(i)}_{\varepsilon} = [\Lambda^{(i)}_{\varepsilon}]^2 \hat{u}^{(i)}_{\varepsilon} \hat{u}^{(i)}_{\varepsilon} + [\Lambda^{(i)}_{\varepsilon}]^{-2} \hat{s}^{(i)}_{\varepsilon} \hat{s}^{(i)}_{\varepsilon}.$$
Perturbation Results

\[ \Lambda^{(i)}_{\varepsilon} = \Lambda^{i} (1 + \varepsilon \eta^{(i)}), \quad \hat{\mathbf{u}}^{(i)}_{\varepsilon} = \hat{\mathbf{u}} + \varepsilon \zeta^{(i)} \hat{\mathbf{s}}, \]

\[ \eta^{(i)} = \frac{1}{2} \sin 2\theta \sum_{j=0}^{i-1} \cos \left( 2\pi (M^{j} \cdot \mathbf{X})_{1} \right); \]

\[ \zeta^{(i)} = \frac{1}{\Lambda^{2i} - \Lambda^{-2i}} (\zeta^{(i)}_{+} + \zeta^{(i)}_{-}), \]

\[ \zeta^{(i)}_{\pm} = \frac{1}{2} (\cos 2\theta \mp 1) \sum_{j=0}^{i-1} \Lambda^{\pm2(i-j)} \cos \left( 2\pi (M^{j} \cdot \mathbf{X})_{1} \right). \]

Observe that the perturbation to the eigenvectors converges exponentially, as required.
Perturbed Metric Tensor

\[ \mathbb{D}^{(i)} = \kappa \left[ g^{(i)}_{\varepsilon} \right]^{-1}; \quad \left[ g^{(i)}_{\varepsilon} \right]^{-1} = \left[ \Lambda^{(i)}_{\varepsilon} \right]^{2} \hat{s}^{(i)} \hat{s}^{(i)} + \left[ \Lambda^{(i)}_{\varepsilon} \right]^{-2} \hat{u}^{(i)} \hat{u}^{(i)}. \]

To leading order in \( \varepsilon \), we have

\[ \left[ g^{(i)}_{\varepsilon} \right]^{-1} = \Lambda^{2i} \hat{s} \hat{s} + \Lambda^{-2i} \hat{u} \hat{u} + 2\varepsilon \eta^{(i)} (\Lambda^{2i} \hat{s} \hat{s} - \Lambda^{-2i} \hat{u} \hat{u}) - \varepsilon \zeta^{(i)} (\Lambda^{2i} - \Lambda^{-2i}) (\hat{u} \hat{s} + \hat{s} \hat{u}), \]

where the only functions of \( X \) are \( \eta^{(i)} \) and \( \zeta^{(i)} \).
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Recall the solution to the A–D equation:

\[ \hat{\theta}^{(i)}_{k} = \sum_{\ell} \exp \left( g^{(i)}_{k\ell} \right) \hat{\theta}^{(i-1)}_{\ell}. \]
The Exponent $G^{(i)}$

\[
G^{(i)}_{k\ell} = -4\pi^2 T \int_{T^2} (\mathbf{k} \cdot \mathbf{D}^{(i)} \cdot \mathbf{l}) e^{-2\pi i (k_0 - \ell_0) \cdot X} d^2 X \\
= A^{(i)}_{k\ell} + \varepsilon B^{(i)}_{k\ell}
\]
The Exponent $\mathcal{G}^{(i)}$

\[ \mathcal{G}_{k\ell}^{(i)} = -4\pi^2 T \int_{\mathbb{T}^2} (k \cdot \mathbb{D}^{(i)} \cdot \ell) \, e^{-2\pi i (k-\ell) \cdot X} \, d^2 X \]

\[ = A_{k\ell}^{(i)} + \varepsilon B_{k\ell}^{(i)} \]

where

\[ A_{k\ell}^{(i)} = -\kappa \left( \Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2 \right) \delta_{k\ell}, \quad \kappa := 4\pi^2 \tilde{\kappa} T \]

\[ B_{k\ell}^{(i)} = -\kappa \left( 2 \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right) \eta_{k\ell}^{(i)} \right. \]

\[ - \left( k_u \ell_s + k_s \ell_u \right) \left( \zeta_{-k\ell}^{(i)} + \zeta_{k\ell}^{(i)} \right) \left. \right) \]

with $k_u := (k \cdot \hat{u})$, $k_s := (k \cdot \hat{s})$. 

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The Exponent $G^{(i)} = A^{(i)} + \varepsilon B^{(i)}$ (cont’d)

The diagonal part, $A^{(i)}$, inexorably leads to superexponential decay of variance, because it grows exponentially. Upon making use of the Fourier-transformed $\zeta^{(i)}$ and $\eta^{(i)}$, we find

$$B_{k\ell}^{(i)} = -\frac{1}{2} \kappa \sum_{j=0}^{i-1} B_{k\ell}^{ij} \left( \delta_{k,\ell+\hat{e}_1 \cdot Mj} + \delta_{k,\ell-\hat{e}_1 \cdot Mj} \right)$$

$$B_{k\ell}^{ij} = \sin 2\theta \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right)$$

$$+ (k_u \ell_s + k_s \ell_u) \left( \Lambda^{2(i-j)} \sin^2 \theta - \Lambda^{-2(i-j)} \cos^2 \theta \right).$$

So $B^{(i)}$ is not diagonal (it couples different modes to each other).

$\longrightarrow$ Dispersive in Fourier space.
But can we Compute the Exponential, $\exp(G^{(i)})$?

To leading order in $\varepsilon$, for $A$ diagonal, we have

$$[\exp(A^{(i)} + \varepsilon B^{(i)})]_{k\ell} = e^{A_{kk}^{(i)}} \delta_{k\ell} + \varepsilon E_{k\ell}^{(i)}; \quad E_{k\ell}^{(i)} = B_{k\ell}^{(i)} \frac{e^{A_{kk}^{(i)}} - e^{A_{\ell\ell}^{(i)}}}{A_{kk}^{(i)} - A_{\ell\ell}^{(i)}}.$$

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- However, the $\Lambda^{2i}$ term in $A_{kk}^{(i)}$ precludes any optimism about the situation: it dooms us to a grim superexponential death.
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- However, the $\Lambda^{2i}$ term in $A_{kk}^{(i)}$ precludes any optimism about the situation: it dooms us to a grim superexponential death.
- For $\varepsilon = 0$, this is indeed what happens. But for a finite value of $\varepsilon$, the $E$ term breaks the diagonality of $\mathcal{G}$, so that given some initial set of wavevectors, the variance contained in those modes can be transferred elsewhere.
A Few Words about Numerics

- Impractical to take the matrix exponential for large matrices.
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• Perturbative expansion sidesteps this problem.

\[ \exp(-i k^2 u s) \]
A Few Words about Numerics

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However, still need to go to extremely high wavenumber . . . impossible to use mesh, since would have to refine exponentially fast.

So keep track of only the required wavevectors: their number should grow exponentially . . . but it doesn’t!

This is because as \( i \) increases, most modes are damped as

\[
\exp \left(-\kappa \left( \Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2 \right) \right),
\]

except for those that have very small \( k_s = (k \cdot \hat{s}) \), i.e., those that are aligned with \( \hat{u} \).
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  \exp\left(-\kappa \left(\Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2\right)\right),
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  except for those that have very small \( k_s = (k \cdot \hat{s}) \), i.e., those that are aligned with \( \hat{u} \).
- Just let computer take care of pruning via underflow!
A Few Words about Numerics

- Impractical to take the matrix exponential for large matrices.
- Perturbative expansion sidesteps this problem.
- However, still need to go to extremely high wavenumber ... impossible to use mesh, since would have to refine exponentially fast.
- So keep track of only the required wavevectors: their number should grow exponentially ... but it doesn’t!
- This is because as $i$ increases, most modes are damped as $\exp \left( -\kappa \left( \Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2 \right) \right)$, except for those that have very small $k_s = (k \cdot \hat{s})$, i.e., those that are aligned with $\hat{u}$.
- Just let computer take care of pruning via underflow!
- The surviving modes need to become more and more aligned with $\hat{u}$ as time goes on.
Comparison: Eulerian and Lagrangian Views

![Graph showing comparison between Eulerian and Lagrangian views.](image)

- **Graph**: Comparison of variance over iterations for different diffusion coefficients ($D=0.1$, $D=0.01$, $D=0.001$, $D=0.0001$) and Lagrangian view. The variance decreases significantly, indicating effective mixing.

- **Key Points**:
  - The plots demonstrate how variance reduces as iterations increase.
  - The diffusion coefficient $D$ significantly affects the rate of variance reduction.

- **Legend**:
  - Green: $D=0.1$
  - Blue: $D=0.01$
  - Purple: $D=0.001$
  - Cyan: $D=0.0001$
  - Red: Lagrangian view

- **Equation**: $\varepsilon = 10^{-4}$

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Eulerian and Lagrangian Pictures of Mixing – p.27/30
iteration $= 4$

$\kappa = 0.01$
Rescaled Pattern for $i = 6, \ldots, 12$

$\varepsilon = 10^{-4}$

$\kappa = 0.1$
Conclusions

- In the Eulerian view, large-scale eigenmode dominates exponential phase, as for baker’s map.
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- It is not possible to simply transform the Eulerian result to Lagrangian coordinates, since orbits are chaotic . . . must solve Lagrangian problem from the start.
- There exists a kind of pattern in Lagrangian coordinates (not eigenfunction) that is cascading to large wavenumbers.
Conclusions

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• Pattern confined to dominant mode in Eulerian coordinates, but dispersed in Lagrangian space.
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• Still some kinks to iron out!