The One-dimensional Nature of the Advection–Diffusion Equation

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with Allen Boozer and David Lazanja
The Advection–Diffusion Equation

\[ \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho D \nabla \phi) \]

Large Péclet number limit: \( Pe = \nu L / D \gg 1 \) is the norm rather than the exception.

Typical values of \( Pe \):

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<table>
<thead>
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<tbody>
<tr>
<td>Earth’s core</td>
<td>(10^3)</td>
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<tr>
<td>Heat in a room</td>
<td>(10^5)</td>
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<tr>
<td>Solar corona</td>
<td>(10^{12})</td>
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<tr>
<td>Galaxy</td>
<td>(10^{19})</td>
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Singular limit: Even a tiny amount of diffusivity matters.
Chaotic Mixing

- Chaotic trajectories of fluid particles generates small scales, even for non-turbulent flows;
- **Huge** gradients of $\phi$ are created;
- Makes enhanced diffusion possible:
  For heat in a room, turns a diffusion time of months into minutes (exponential)
- Very difficult to simulate directly: scale separation $\sim 10^{10}$;
- **Lagrangian** (comoving) coordinates are very convenient because the chaos gets “hidden” in the coordinate transformation.
- The study of Lagrangian quantities leads to some surprising results: they obey constraints due to the chaotic nature of the flow, which leads to a one-dimensional diffusion equation in Lagrangian coordinates.
In Lagrangian coordinates $\mathbf{a}$, the advection-diffusion equation is

$$\frac{\partial \phi}{\partial t} \bigg|_{\mathbf{a}} = \sum_{p,q} D \frac{\partial}{\partial a^p} \left[ g^{pq} \frac{\partial \phi}{\partial a^q} \right]$$

where $g^{pq} = (g^{-1})^{pq}$ characterizes the transformation from Eulerian to Lagrangian coordinates. Assuming the flow is chaotic, can approximate by

$$\frac{\partial \phi}{\partial t} \bigg|_{\mathbf{a}} = \sum_{p,q} D \frac{\partial}{\partial a^p} \left[ \Lambda_s^{-2} \hat{s}^p \hat{s}^q \frac{\partial \phi}{\partial a^q} \right]$$

where $\Lambda_s(\mathbf{a}, t) \ll 1$ is a coefficient of expansion associated with the contracting direction $\hat{s}(\mathbf{a}, t)$ of the flow.

Not quite a 1–D diffusion equation...
A One-dimensional Equation

Chaotic flows satisfy several differential constraints, one of which is

$$\sum_p \left( \frac{\partial}{\partial a_p} \hat{s}^p - \hat{s}^p \frac{\partial}{\partial a_p} \log \Lambda_s \right) \to 0,$$

where \( \hat{s} \) denotes any contracting direction.


Using this constraint, we find a one-dimensional diffusion equation

$$\left. \frac{\partial \phi}{\partial t} \right|_a = \tilde{D}(t) \frac{\partial^2 \phi}{\partial s^2}$$

where

$$\frac{\partial}{\partial s} \equiv \sum_p \tilde{\Lambda}_s^{-1} \hat{s}^p \frac{\partial}{\partial a_p}$$

Exponentially-growing diffusion coefficient, \( \tilde{D}(t) \).

Variation in mixing along manifold given by the function \( \tilde{\Lambda}_s(a) \).
\( \tilde{\Lambda}_s \) is large whenever the curvature is large
\[ \Rightarrow \text{Suppression of stretching.} \quad [\text{Drummond and Münch (1991)}] \]
Stretching vs Curvature along a Material Line
Cellular Flow

\[ \log_\nu = 3 \]

\[ \kappa^{1/3} \]

\[ \log \tilde{A}_s \]

\[ \log \kappa \]
Material Line Folded by a Flow

- Assume linear gradient of $\phi$ varying from 0 to 1;
- The endpoints of the line are brought to a distance $\Delta$;
- Enhancement in $\nabla \phi$ proportional to $\Delta^{-1}$;
- Points in the crest of the bend do not benefit.
The “folding” model predicts the $\tilde{\Lambda}_s^{-2}$ tail of the probability of extremely low stretching events. Exponential ("fat") tail: can have a tremendous impact on mixing.
Stationary distribution. Tails seem independent of specific flow.
In the small diffusivity limit, the advection–diffusion equation is difficult to solve directly because of small scales.

Reduce the advection–diffusion equation to one dimension in Lagrangian coordinates. [Thiffeault, Physics Review E (2002)]

Requires the use of differential constraints [Thiffeault and Boozer, Chaos (2001); Thiffeault, Nonlinearity (2002)].

In smooth flows, important to understand the detailed manner in which gradients are enhanced through folding.

Simple model captures the behavior at sharp bends.