Stretching and Curvature in Chaotic Flows

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Material Lines in Flows

How do material lines embedded in a chaotic flow evolve?

⇒ Stretch, Twist, Fold

Relevance:

• **Magnetic dynamo**: evolution of magnetic field in a plasma.

• **Chemical and biological mixing**: creation of intermaterial contact area.

• **Polymer mixing** (i.e., DNA): follow material lines closely.

• Much is known about **stretching**, but less about the bending of material lines (generation of **curvature** and **torsion**).

Some interesting regularities, such as a close **anticorrelation** between stretching and curvature.
Stretching and Folding

Traces out the unstable foliation of the flow. Note the sharp folds that develop.

Cellular Flow

Standard Map

Can look surprisingly regular even in extremely chaotic cases.
\( \tilde{\Lambda} \) is the deviation from mean stretching.

⇒ Suppression of stretching.  [Drummond & Münch, JFM 225, 529 (1991)]
A similar effect was recently observed for the magnetic dynamo.

The magnetic field and its curvature are anticorrelated

Power law relation around sharp folds: The "−1/3" law.

The law is very robust even with varying degree of chaos and different flows (2D and 3D).
A Foliation of Bends

Some observations:

- Material lines are not isolated objects.
- Continuum of other material lines.
- Standard map resembles a foliation of bends.
- Distance between lines is not constant: Compression is not uniform.
- Curvature is readily computed (geometrical).
- How do we relate to stretching?
Conservation Law for Lyapunov Exponents

The tangent to the material line aligns with the **unstable direction** of the flow, \( \hat{u} \), the direction of maximum stretching. That direction satisfies the crucial **constraint**

\[
\nabla \cdot \hat{u} + \hat{u} \cdot \nabla \log \tilde{\Lambda} \longrightarrow 0, \quad \text{(exponentially)}
\]


This is a conservation law on for \( \tilde{\Lambda} \) along the unstable manifold.

\[
\frac{\partial}{\partial \tau} \log \tilde{\Lambda} + \nabla \cdot \hat{u} = 0, \quad \tau \equiv \text{arc length along } \hat{u}
\]

**Convergence of } \hat{u} \Rightarrow \text{ increase in } \tilde{\Lambda}.\]
Assuming a foliation of bends with shape \( y = f(x) \), the divergence of \( \hat{u} \) is easily computed,

\[
\nabla \cdot \hat{u} \simeq \frac{\partial \hat{u}_x}{\partial x} = -\frac{f'f''}{(1 + f'^2)^{3/2}}.
\]

Derivative of \( \tilde{\Lambda} \) along \( \hat{u} \):

\[
\frac{\partial}{\partial \tau} \log \tilde{\Lambda} = \hat{u} \cdot \nabla \log \tilde{\Lambda} = \frac{1}{(1 + f'^2)^{1/2}} \frac{\partial}{\partial x} \log \tilde{\Lambda},
\]

Equate and integrate to yield

\[
\tilde{\Lambda} \sim (1 + f'^2)^{1/2}.
\]
To exhibit the relationship between stretching and curvature, we use

\[ \kappa(x) = \frac{|f''(x)|}{(1 + f'^2)^{3/2}} \]

for the magnitude of the curvature and obtain finally

\[ \tilde{\Lambda} \sim |f''(x)|^{1/3} \kappa^{-1/3} \]

For quadratic \( f \),

\[ \tilde{\Lambda} \sim (\kappa/\kappa_0)^{-1/3}, \]

so that the power-law relation holds exactly.

The shape of the bend and \( y \)-dependence of the tangent vector field will cause deviations from the \(-1/3\) law.
The “folding” model predicts the $\tilde{\Lambda}^2$ tail of the probability of extremely low stretching events. Exponential (“fat”) tail: large fluctuations from the mean stretching.
Stationary distribution. Tails seem independent of specific flow. Mean moves to the right in less chaotic flows.
Conclusions

• Stretching **anticorrelated** with curvature.

• Around sharp bends, observe **stretching \( \sim \) curvature\(^{-1/3}\).**

• Can be explained using a **conservation law** for Lyapunov exponents.

Ongoing work:

• The consequences of **constraints** in physical applications (for the **dynamo** [JLT & Boozer, Physics of Plasmas **10** (2003)]).

• Evolution of **torsion**. Constrained, like curvature?

• Understand PDF of curvature. 2D special?