Topological detection of Lagrangian coherent structures

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How do we efficiently detect trajectories that ‘bunch’ together?

This is the central problem for the detection of barriers to transport, or Lagrangian coherent structures (LCS).

[movie 1]
Growth of curves with moving obstacles

With 3 obstacles (floats), we can also look at the growth of curves:

The motion above is denoted $\sigma_1 \sigma_2^{-1}$.

The rate of growth of the loop is called the topological entropy.
Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;

2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.
Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:
Action on coordinates

Moving the obstacles changes some crossing numbers:

There is an explicit formula for the change in the coordinates! (Dynnikov, 2002; Moussafir, 2006; Thiffeault, 2010)
Growth of loop length

For a specific rod motion, we can easily see the exponential growth of $L$ and thus measure the entropy:
Oceanic float trajectories
Oceanic floats: Entropy

10 floats from Davis’ Labrador sea data:

Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)
Lagrangian Coherent Structures

- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not ‘leaky.’
Growth of a vast number of loops

Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops: fast and slow-growing.
What do the slowest-growing loops look like?

The slowest-growing loops surround bunches of trajectories that travel together (remain in the same ergodic component):

[(a) appears because the coordinates also encode ‘multiloops.’]
A physical example: Rod stirring device
Conclusions

- Chaotic trajectories undergo ‘braiding’ motion that leads to growth of ‘topological loops.’ (crude Lyapunov exponent)
- Need a way to compute entropy fast: loop coordinates;
- There is a lot more information in this braid: extract invariant regions (related to Lagrangian coherent structures);
- Currently refining the technique, and applying to float data in the ocean as well as granular particle data (with K. Daniels, J. Puckett, and F. Lechenault).
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References


