The Energy-Casimir Method and MHD Stability

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Overview

- Many equations of plasma physics have a Hamiltonian formulation in terms of Lie–Poisson brackets.

- We investigate the structure of these Lie–Poisson brackets. The simplest case is the semidirect sum structure.

- Some systems, such as a model of 2-D compressible reduced MHD, have a more complicated structure involving cocycles.

- We look at the role of cocycles in formal stability. The principle is similar that of $\delta W$ energy methods, but we determine stability criteria using the concept of dynamical accessibility, which uses the bracket directly. This is closely related to the energy-Casimir method.
Hamiltonian Formulation

A system of equations has a Hamiltonian formulation if it can be written in the form

$$\dot{\xi}^\lambda(x, t) = \{\xi^\lambda, H\}$$

where $H$ is a Hamiltonian functional, and $\xi(x)$ represents a vector of field variables (vorticity, temperature, \ldots ).

The Poisson bracket $\{,\}$ is antisymmetric and satisfies the Jacobi identity,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

This tells us that there exist local canonical coordinates.
Lie–Poisson Brackets

A particular type of bracket is the Lie–Poisson bracket,

\[
\{F, G\} = \int_\Omega W_\lambda^{\mu\nu} \xi^\lambda(x, t) \left[ \frac{\delta F}{\delta \xi^\mu(x, t)}, \frac{\delta G}{\delta \xi^\nu(x, t)} \right] d^2x
\]

where repeated indices are summed, and \( x = (x, y) \). The 3-tensor \( W \) is constant, and determines the structure of the bracket. The inner bracket is the 2-D Jacobian,

\[
[a, b] = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial a}{\partial y}.
\]

The 2-D fluid domain is denoted by \( \Omega \).
Example: Compressible Reduced MHD

The four-field model derived by Hazeltine et al. [7] for 2-D compressible reduced MHD (CRMHD) has a Lie–Poisson structure. The model includes compressibility and finite ion Larmor radius effects. The field variables are

\[ \omega \quad \text{vorticity} \]
\[ v \quad \text{parallel velocity} \]
\[ p \quad \text{pressure} \]
\[ \psi \quad \text{magnetic flux} \]

and are functions of \((x, y, t)\). There is also a constant parameter \(\beta_i\) that measures compressibility.
The equations of motion for CRMHD are

\[ \begin{align*} 
\dot{\omega} &= [\omega, \phi] + [\psi, J] + 2 [p, x] \\
\dot{v} &= [v, \phi] + [\psi, p] + 2\beta_i [x, \psi] \\
\dot{p} &= [p, \phi] + \beta_i [\psi, v] \\
\dot{\psi} &= [\psi, \phi], 
\end{align*} \]

where \( \omega = \nabla^2 \phi \), \( \phi \) is the electric potential, \( \psi \) is the magnetic flux, and \( J = \nabla^2 \psi \) is the current.

The Hamiltonian is just the energy,

\[
H[\omega, v, p, \psi] = \frac{1}{2} \int_\Omega \left( |\nabla \phi|^2 + v^2 + \frac{(p - 2\beta_i x)^2}{\beta_i} + |\nabla \psi|^2 \right) \, d^2 x.
\]
The equations for CRMHD are obtained by inserting the above Hamiltonian into the Lie–Poisson bracket

$$\{A, B\} = \int_{\Omega} \left( \omega \left[ \frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta \omega} \right] + v \left( \left[ \frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta v} \right] + \left[ \frac{\delta A}{\delta v}, \frac{\delta B}{\delta \omega} \right] \right) + p \left( \left[ \frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta p} \right] + \left[ \frac{\delta A}{\delta p}, \frac{\delta B}{\delta \omega} \right] \right) + \psi \left( \left[ \frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta \psi} \right] + \left[ \frac{\delta A}{\delta \psi}, \frac{\delta B}{\delta \omega} \right] \right) \right) - \beta \psi \left( \left[ \frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta v} \right] + \left[ \frac{\delta A}{\delta v}, \frac{\delta B}{\delta \omega} \right] \right) \right) d^2x.$$

This can be shown to satisfy the Jacobi identity. Comparing this to our definition of the Lie–Poisson bracket with the definition $$(\xi^0, \xi^1, \xi^2, \xi^3) = (\omega, v, p, \psi)$$, we can read off the tensor $W$. 
The $W$ tensor for CRMHD

Since $W$ is a 3-tensor, we can represent it as a cube:

The small blocks denote nonzero entries. The “shape” of $W$ is constrained by the Jacobi identity. The vertical axis is the lower index of $W_\lambda^{\mu\nu}$, with the origin at the top rear. The two horizontal axes are the symmetric upper indices.
Semidirect Sums

A common form for the bracket is the semidirect sum (SDS), for which $W$ looks like the picture below.

Note that CRMHD does not have a semidirect sum structure because of its extra nonzero blocks. These extra blocks, proportional to $\beta_i$, are called cocycles.

Reduced MHD (two fields: $\omega$ and $\psi$) has a semidirect sum structure. This structure arises in systems with a vorticity-like field variable that advects the other quantities of the model.
Casimir Invariants

Noncanonical brackets can have Casimir invariants, which are functionals $C$ which commute with every other functional:

$$\{ F, C \} \equiv 0, \quad \text{for all } F.$$  

Casimirs are conserved quantities for any Hamiltonian.

For CRMHD, they are

\[
\begin{align*}
C^0 &= \int_\Omega \left( \omega f(\psi) - \frac{1}{\beta_i} p_v f'(\psi) \right) \, d^2x, \\
C^1 &= \int_\Omega v g(\psi) \, d^2x, \\
C^2 &= \int_\Omega p h(\psi) \, d^2x, \\
C^3 &= \int_\Omega k(\psi) \, d^2x.
\end{align*}
\]
The Energy-Casimir Method

Requiring that a solution $\xi_e$ be a constrained minimum of the Hamiltonian,

$$\delta(H + C)[\xi_e] =: \delta F[\xi_e] = 0,$$

gives an equilibrium solution. The solutions $\xi_e$ is then said to be formally stable if $\delta^2 F[\xi_e]$ is definite. This is related to $\delta W$ energy principles, which extremize the potential energy.
Dynamical Accessibility

A slightly more general method for establishing formal stability uses dynamically accessible variations (DAV), defined as

$$\delta \xi_{da} := \{G, \xi\} + \frac{1}{2} \{G, \{G, \xi\}\},$$

with $G$ given in terms of the generating functions $\chi_\mu$ by

$$G := \int_\Omega \xi^\mu \chi_\mu \, d^2x.$$ 

DAV are variations that are constrained to remain on the symplectic leaves of the system. They preserve the Casimirs to second order. Stationary solutions of the Hamiltonian,

$$\delta H_{da}[\xi_e] = 0,$$

capture all possible equilibria of the equations of motion.
Energy Associated with DAVs

The energy of the perturbations is

$$\delta^2 H_{da}[\xi_e] = \frac{1}{2} \int_{\Omega} \left( \delta \xi_{da} \frac{\delta^2 H}{\delta \xi_\sigma \delta \xi_\tau} \delta \xi_{da}^\tau - W^{\mu\nu}_\chi \delta \xi_{da}^\lambda \left[ \chi_\mu , \frac{\delta H}{\delta \xi_\nu} \right] \right) d^2x.$$

In order to determine sufficient conditions for stability, we need to write $\delta^2 H_{da}$ in terms of the $\delta \xi_{da}^\lambda$ only (no explicit $\chi_\mu$ dependence). In principle, this can always be done.
Equilibrium Solutions of Semidirect Sums

An equilibrium \((\omega_e, \{\xi_e^\mu\})\) of the equations of motion for an SDS satisfies

\[
\dot{\omega}_e = \left[ \frac{\delta H}{\delta \xi^0}, \omega_e \right] + \sum_{\mu=1}^{n} \left[ \frac{\delta H}{\delta \xi^\mu}, \xi_e^\mu \right] = 0,
\]

\[
\dot{\xi}_e^\mu = \left[ \frac{\delta H}{\delta \xi^0}, \xi_e^\mu \right] = 0, \quad \mu = 1, \ldots, n,
\]

where we have labeled the 0th variable by \(\omega\). We can satisfy the \(\dot{\xi}_e^\mu = 0\) equations by letting

\[
\frac{\delta H}{\delta \xi^0} = -\Phi(u), \quad \xi_e^\mu = \Xi^\mu(u), \quad \mu = 1, \ldots, n,
\]

for arbitrary functions \(u(x)\), \(\Phi(u)\), and \(\Xi^\mu(u)\).
DAVs for Semidirect Sums

The dynamically accessible variations for an SDS are

\[ \delta \omega_{da} = [\omega, \chi_0] + \sum_{\nu=1}^{n} [\xi^\nu, \chi_\nu], \]

\[ \delta \xi^\mu_{da} = [\xi^\mu, \chi_0], \quad \mu = 1, \ldots, n. \]

Notice how all the \( \delta \xi^\mu_{da} \) depend only on \( \chi_0 \): the only allowed perturbations are rearrangements of the vorticity \( \omega \).
CRMHD Equilibria

An equilibrium of Equations (1) satisfies

\[ \psi_e = \Psi(u), \]
\[ \phi_e = \Phi(u), \]
\[ v_e = \left( k_1(u) + (k_2(u) + 2x) \Phi'(u) \right) / \left( 1 - |\Phi'(u)|^2 / \beta_i \right), \]
\[ p_e = \left( k_1(u) \Phi'(u) + \beta_i (k_2(u) + 2x) \right) / \left( 1 - |\Phi'(u)|^2 / \beta_i \right), \]
\[ \omega_e \Phi'(u) - J_e = k_3(u) + v_e k_1'(u) + p_e k_2'(u) + \beta_i^{-1} p_e v_e \Phi''(u), \]

with primes defined by \( f'(u) = (d\Psi(u)/du)^{-1} df(u)/du \), and \( u(x), \Psi(u), \Phi(u), \) and the \( k_i(u) \) arbitrary functions.

This is very different from the SDS case. In particular, the cocycle allows the equilibrium “advected” quantities \( v_e \) and \( p_e \) to depend explicitly on \( x \).
DAVs for CRMHD

The dynamically accessible variations for CRMHD are given by

$$\delta \omega_{da} = [\omega, \chi_0] + [v, \chi_1] + [p, \chi_2] + [\psi, \chi_3],$$
$$\delta v_{da} = [v, \chi_0] - \beta_i [\psi, \chi_2],$$
$$\delta p_{da} = [p, \chi_0] - \beta_i [\psi, \chi_1],$$
$$\delta \psi_{da} = [\psi, \chi_0].$$

Note that the DAV for $\omega$ is the same as for a semidirect sum. However, the “advected” quantities $v$, $p$, and $\psi$ now have independent variations, which can be specified by $\chi_2$, $\chi_1$, and $\chi_0$, respectively.
CRMHD Stability

The terms that involve gradients in the perturbation energy are

\[ \delta^2 \mathcal{H}_{\text{da}} = \int_{\Omega} \left( |\nabla \delta \phi - \nabla (\Phi'(u) \delta \psi)|^2 
+ (1 - |\Phi'(u)|^2)|\nabla \delta \psi|^2 + \cdots \right) \, d^2 x. \]

These terms must be positive, so we require

\[ |\Phi'(u)| < 1, \quad \text{(2)} \]

part of the sufficient condition for stability.
The remaining terms are a quadratic form in $\delta v_{da}$, $\delta p_{da}$, and $\delta \psi_{da}$, which can be written

$$
\begin{pmatrix}
1 & -\beta_1^{-1} \Phi' & -k_1' - \beta_1^{-1} p_e \Phi'' \\
-\beta_1^{-1} \Phi' & \beta_1^{-1} & -k_2' - \beta_1^{-1} v_e \Phi'' \\
-k_1' - \beta_1^{-1} p_e \Phi'' & -k_2' - \beta_1^{-1} v_e \Phi'' & \Theta(x, y)
\end{pmatrix}
$$

where

$$
\Theta(x, y) := -k_3'(u) - v_e k_1''(u) - p_e k_2''(u) + \omega_e \Phi''(u) - \beta_1^{-1} p_e v_e \Phi'''(u) + \Phi'(u) \nabla^2 \Phi'(u).
$$

For positive-definiteness of this quadratic form, we require the principal minors of this matrix to be positive.
\[ \mu_1 = |1| > 0, \]
\[ \mu_2 = \begin{vmatrix} 1 & -\beta_i^{-1} \Phi'(u) \\ -\beta_i^{-1} \Phi'(u) & \beta_i^{-1} \end{vmatrix} = \beta_i^{-1} \left( 1 - \frac{\Phi'(u)^2}{\beta_i} \right) > 0, \]

The positive-definiteness of \( \mu_2 \), combined with condition (2), implies

\[ |\Phi'(\psi_e)|^2 < \min(1, \beta_i) \]

which is part of a sufficient condition for stability. Thus the cocycle modifies the stability directly: it always makes the stability condition \textit{worse}, because \( \beta_i > 0 \).

Finally, if we require that the \textit{determinant} of the matrix be positive, we have a sufficient condition for formal stability.
Conclusions

- The Lie–Poisson structure of a system tells us the form of the perturbations allowed by the constraints.

- These perturbations can be used to establish sufficient conditions for formal stability.

- Equilibrium solutions for semidirect sums involve advected quantities that are tied to the fluid elements. When a cocycle is present in the bracket, such as for CRMHD, the equilibria are richer.

- In the case of CRMHD, the cocycle has a destabilizing effect on the system, as compared to a semidirect sum structure.
References


