Curvature in Chaotic and Turbulent Flows

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How do material lines embedded in a chaotic flow evolve?

⇒ Stretch, Twist, Fold

Relevance:

• **Magnetic dynamo**: evolution of magnetic field in a plasma.
• **Chemical and biological mixing**: creation of intermaterial contact area.
• **Polymer mixing** (*i.e.*, DNA): follow material lines closely.
• Much is known about **stretching**, but less about the bending of material lines (generation of **curvature** and **torsion**).

Some interesting regularities, such as a close **anticorrelation** between stretching and curvature.
Stretching and Folding

Traces out the **unstable foliation** of the flow.
Note the **sharp folds** that develop.

![Cellular Flow](image1)

![Standard Map](image2)

Can look surprisingly **regular** even in **extremely chaotic cases**.
Stretching along a Material Line

\( \tilde{\Lambda} \) is the deviation from mean stretching.

\( \tilde{\Lambda} \)

Cellular Flow

Standard Map

⇒ Suppression of stretching. [Drummond & Münch (1991)]
A similar effect was recently observed for the magnetic dynamo.

The magnetic field and its curvature are **anticorrelated**

Power law relation around sharp folds: The \( "-1/3" \) law.

The law is very **robust** even with varying degree of chaos and different flows (2D and 3D).
Enhancement to Gradients by Folding

- Assume linear gradient of $\phi$ varying from 0 to 1;
- The endpoints of the line are brought to a distance $\delta$;
- Enhancement in $\nabla \phi$ proportional to $\delta^{-1}$;
- Fluid elements in the crest of the bend do not benefit.
- Can explain $-1/3$ law with this simple model. [JLT, 2002]
A Simple Model

Very sharp bend in a material line,

\[ y = f(x) = \frac{1}{2} \kappa_0 x^2 + O(x^3) \]

where \( \kappa_0 = f''(0) \) is the curvature at the tip. \( f(x) \gg x \) away from the tip. Approximate the arc length \( \tau \) from \((0, 0)\) to \((x, f(x))\) by

\[ \tau(x) \simeq f(x). \]

Enhancement to gradients:

\[ \tilde{\Lambda}(x) = \tau(x)/x \simeq f(x)/x. \]

⇒ Measure of stretching (incompressible)
The curvature is $\kappa \equiv |(\hat{t} \cdot \nabla)\hat{t}|$, where $\hat{t}$ is the unit tangent to $f$. To leading order this is

$$\kappa(x) = \kappa_0^{-2} x^{-3} + O(x^{-2}), \quad \tilde{\Lambda}(x) = \kappa_0 x + O(x^2).$$

Solve for $x$ in terms of $\kappa$,

$$\tilde{\Lambda} \sim \kappa^{-1/3}$$

**Problem:** the $-1/3$ law works **much better** than predicted by this simple model.

(Predicts breakdown near the tip, works fine in 3D...)

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A Foliation of Bends

Some observations:

- Material lines are not isolated objects.
- Continuum of other material lines.
- Standard map resembles a foliation of bends.
- Distance between lines is not constant: Compression is not uniform.
- Curvature is readily computed (geometrical).
- How do we relate to stretching?
Conservation Law for Lyapunov Exponents

The tangent to the material line aligns with the unstable direction of the flow, \( \hat{u} \), the direction of maximum stretching. That direction satisfies the crucial constraint

\[
\nabla \cdot \hat{u} + \hat{u} \cdot \nabla \log \tilde{\Lambda} \rightarrow 0, \quad \text{(exponentially)}
\]


This is a conservation law on for \( \tilde{\Lambda} \) along the unstable manifold.

\[
\frac{\partial}{\partial \tau} \log \tilde{\Lambda} + \nabla \cdot \hat{u} = 0, \quad \tau \equiv \text{arc length along } \hat{u}
\]

Convergence of \( \hat{u} \) \( \Rightarrow \) increase in \( \tilde{\Lambda} \).
Assuming a foliation of bends with shape $y = f(x)$, the divergence of $\hat{u}$ is easily computed,

$$\nabla \cdot \hat{u} \simeq \nabla \cdot \hat{t} = \frac{\partial t_x}{\partial x} = -\frac{f' f''}{(1 + f'r^2)^{3/2}}.$$

Derivative of $\tilde{\Lambda}$ along $\hat{u}$:

$$\frac{\partial}{\partial \tau} \log \tilde{\Lambda} = \hat{u} \cdot \nabla \log \tilde{\Lambda} = \frac{1}{(1 + f'r^2)^{1/2}} \frac{\partial}{\partial x} \log \tilde{\Lambda},$$

Equate and integrate to yield

$$\tilde{\Lambda} \sim (1 + f'r^2)^{1/2}.$$
To exhibit the relationship between stretching and curvature, we use

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'^2)^{3/2}}$$

for the magnitude of the curvature and obtain finally

$$\tilde{\Lambda} \sim |f''(x)|^{1/3} \kappa^{-1/3}$$

For quadratic $f$,

$$\tilde{\Lambda} \sim \left(\frac{\kappa}{\kappa_0}\right)^{-1/3},$$

so that the power-law relation holds exactly.

The shape of the bend and $y$-dependence of the tangent vector field will cause deviations from the $-1/3$ law.
The “folding” model predicts the $\tilde{\Lambda}^2$ tail of the probability of extremely low stretching events. Exponential (“fat”) tail: large fluctuations from the mean stretching.
Stationary distribution. Tails seem independent of specific flow. Mean moves to the right in less chaotic flows.
Conclusions

- Stretching anticonrelated with curvature.
- Around sharp bends, observe stretching \(\sim \text{curvature}^{-1/3}\).
- Can be explained using a conservation law for Lyapunov exponents.

Ongoing work:

- The consequences of constraints in physical applications (for the dynamo, with A. Boozer).
- Evolution of torsion. Constrained, like curvature?
- Understand PDF of curvature. 2D special?