Knitting and Mixing

Jean-Luc Thiffeault

http://www.ma.imperial.ac.uk/~jeanluc

Department of Mathematics
Imperial College London
Experiment of Boyland *et al.*

\[ \sigma_1 \quad \text{and} \quad \sigma_2 \]

\[ \sigma_1^{-1} \quad \text{and} \quad \sigma_2^{-1} \]

\( \sigma_1 \) and \( \sigma_2 \) are referred to as the generators of the 3-braid group.
Two Stirring Protocols

$\sigma_1\sigma_2$ protocol

$\sigma_1^{-1}\sigma_2$ protocol

σ₁σ₂ protocol

σ⁻¹₁σ₂ protocol

Let $I$ and $II$ denote the lengths of the two segments. After a $\sigma_2$ operation, we have

\[
\begin{pmatrix}
I' \\
II'
\end{pmatrix} = \begin{pmatrix}
I + II \\
II
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I \\
II
\end{pmatrix} = \sigma_2 \begin{pmatrix}
I \\
II
\end{pmatrix}.
\]

Hence, the matrix representation for $\sigma_2$ is

\[
\sigma_2 = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]
Similarly, after a $\sigma_1^{-1}$ operation we have

$$\begin{pmatrix} I' \\ II' \end{pmatrix} = \begin{pmatrix} I & I + II \\ II & II \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} I \\ II \end{pmatrix} = \sigma_1^{-1} \begin{pmatrix} I \\ II \end{pmatrix}.$$ 

Hence, the matrix representation for $\sigma_1^{-1}$ is

$$\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
Matrix Representation of the Braid Group

We now invoke the faithfulness of the representation to complete the set,

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \sigma_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Our two protocols have representation

\[
\sigma_1 \sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad \sigma_1^{-1} \sigma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]
The Difference between the Protocols

- Each matrix in the group has unit eigenvalues.
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- The first stirring protocol has eigenvalues on the unit circle

So for the second protocol the length of the lines I and II grows exponentially!

The larger eigenvalue is a lower bound on the growth factor of the length of material lines. That is, material lines have to stretch by at least a factor of $2^{1/6180}$ each time we execute the protocol $1^12^2$.

This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).
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Freely-moving Rods in a Cavity Flow

[A. Vikhansky, Physics of Fluids 15, 1830 (2003)]
Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

Material lines must bend around the orbit: it acts just like a “rod”!
The idea: pick any three fluid particles and follow them.

How do they braid around each other?
In the second case there is no net braid: the two elements cancel each other.
We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \cdots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and $N$ is the number of braiding events detected after a time $t$. 

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Random matrix theory says that one eigenvalue of $\Sigma^{(N)}$ can grow exponentially! We call the rate of exponential growth the braiding Lyapunov exponent or just braiding exponent.

The exponent is a measure of the complexity of the braiding motion.
Non-braiding Motion

First consider the motion of three points in concentric circles with irrational frequencies.

The eigenvalue grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.
Blinking-vortex Flow

To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won’t do).

Aref’s **blinking-vortex flow** is ideal.

The only parameter is the circulation $\Gamma$ of the vortices.
For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.

One of the orbits is chaotic, the other two are closed.
For $\Gamma = 13$, the blinking vortex is globally chaotic.

The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order $10^{20}$ rather than 80!
Averaging over many Triplets

\[
\text{slope} = 0.187 \\
\Gamma = 13
\]

Averaged over 100 random triplets.
Comparison with Lyapunov Exponents

\[ \Gamma \text{ varies from 8 to 20.} \]
Conclusions

• Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.
• The complexity of a braid can be represented by the largest eigenvalue of a product of matrices.
• Any triplet of particles can potentially braid.
• The complexity of the braid is a good measure of chaos.
• No need for infinitesimal separation of trajectories or derivatives of the velocity field.
• For instance, can use all the floats in a data set (J. La Casce).
• Test in 2D turbulent simulations (F. Paparella).
• Higher-order braids!