Topological detection of Lagrangian coherent structures

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Sparse trajectories and material loops

How do we efficiently detect trajectories that ‘bunch’ together?

[movie 1]
Mathematical background: Punctured disks

Low-dimensional topologists have long studied transformations of surfaces such as the punctured disk:

![Punctured Disk](image)

The central object of study is the homeomorphism: a continuous, invertible transformation whose inverse is also continuous.

For instance, this is a model of a two-dimensional vat of viscous fluid with stirring rods.
Punctured disks in experiments

The transformation in this case is given by the solution of a fluid equation over one period of rod motion.


[movie 2]  [movie 3]
Growth of curves on a disk

On a disk with 3 punctures (rods), we can also look at the growth of curves:

We use the braid generator notation: $\sigma_i$ means the clockwise interchange of the $i$th and $(i + 1)$th rod. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1\sigma_2^{-1}$. 

\[
\sigma_1 \quad \sigma_1\sigma_2^{-1} \quad \sigma_1\sigma_2 \sigma_1^{-1} \quad \sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}
\]
Growth of curves on a disk (2)

The rate of growth \( h = \log \lambda \) is called the topological entropy.

But how do we find the rate of growth of curves for motions on the disk?

For 3 punctures it’s easy: the entropy for \( \sigma_1 \sigma_2^{-1} \) is \( h = \log \varphi^2 \), where \( \varphi \) is the Golden Ratio!

For more punctures, use Moussafir iterative technique (2006).

Iterating a loop

It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them **topologically** with very few numbers.
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the Dynnikov coordinates involve intersections with vertical lines:
Crossing numbers

Label the crossing numbers:
Dynnikov coordinates

Now take the difference of crossing numbers:

\[ a_i = \frac{1}{2} (\mu_{2i} - \mu_{2i-1}) , \]
\[ b_i = \frac{1}{2} (\nu_i - \nu_{i+1}) \]

for \( i = 1, \ldots, n - 2 \).

The vector of length \((2n - 4)\),

\[ \mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2}) \]

is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can’t do it with fewer than \(2n - 4\) numbers.
Intersection number

A useful formula gives the minimum intersection number with the ‘horizontal axis’:

\[ L(u) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i|, \]

For example, the loop on the left has \( L = 12 \).

The crossing number grows proportionally to the length.
Solution to problem 2: Action on coordinates

Moving the punctures according to a braid generator changes some crossing numbers:

There is an explicit formula for the change in the coordinates!
Action on loop coordinates

The update rules for $\sigma_i$ acting on a loop with coordinates $(a, b)$ can be written

$$a'_{i-1} = a_{i-1} - b'_{i-1} - (b^+_{i-1} + c_{i-1})^+,$$

$$b'_{i-1} = b_i + c^-_{i-1},$$

$$a'_i = a_i - b^-_i - (b^-_{i-1} - c_{i-1})^-,$$

$$b'_i = b_{i-1} - c^-_{i-1},$$

where

$$f^+ := \max(f, 0), \quad f^- := \min(f, 0).$$

$$c_{i-1} := a_{i-1} - a_i - b^+_i + b^-_{i-1}.$$  

This is called a piecewise-linear action. Easy to code up (see for example Thiffeault (2010)).
Growth of $L$

For a specific rod motion, say as given by the braid $\sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_1$, we can easily see the exponential growth of $L$ and thus measure the entropy:
$m$ is the number of times the braid acted on the initial loop.
Oceanic float trajectories
Oceanic floats: Data analysis

What can we measure?

- Single-particle dispersion (not a good use of all data)
- Correlation functions (what do they mean?)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the $\sigma_i$ for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a **topological entropy** for the motion (similar to Lyapunov exponent).
Oceanic floats: Entropy

10 floats from Davis’ Labrador sea data:

Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)
Lagrangian Coherent Structures

- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not ‘leaky.’
Sample system: Modified Duffing oscillator

\[ \dot{x} = y + \alpha \cos \omega t, \]

\[ \dot{y} = x(1 - x^2) + \gamma \cos \omega t - \delta y, \]

+ rotation to further hide two regions. \( \alpha = .1, \gamma = .14, \delta = .08, \omega = 1. \)
Growth of a vast number of loops

Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops!
What do the slowest-growing loops look like?

[(c) appears because the coordinates also encode ‘multiloops.’]
Computational complexity

Here’s the bad news:

- There are an infinite number of loops to consider.
- But we don’t really expect hyper-convoluted initial loops (nor do we care so much about those).
- Even if we limit ourselves to loops with Dynnikov coordinates between $-1$ and $1$, this is still $3^{2n-4}$ loops.
- This is too many... can only treat about 10–11 trajectories using this direct method.
An improved method: Pair-loops

The biggest problem is that we only look at whether a loop grows or not. But there is a lot more information to be found in how a loop entangles the punctures as it evolves.

Consider loops that enclose two punctures at once. More involved analysis, but scales much better with $n$. 

{(1,2), (1,3), (4,2), (4,5)}

{1, 2, 3, 4, 5} {1, 3} {2, 4, 5} {2, 4, 5}
Improvement

Run times in seconds:

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<th># of trajectories</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>20</th>
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<td>128</td>
</tr>
</tbody>
</table>

Bottleneck for the pair-loop method is finding the non-growing loops. (Should scale as $n^2$ for large enough $n$.)

The downside is that the pair-loop method is much more complicated. But in the end it accomplishes the same thing.
A physical example: Rod stirring device

[movie 4]
Conclusions

• Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines);
• This idea can also be used on fluid particles to estimate entropy;
• Need a way to compute entropy fast: loop coordinates;
• There is a lot more information in this braid: extract it! (Lagrangian coherent structures);
• Is this useful? We need a good physical problem to try it on!
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References


