OUTER MODEL THEORY AND THE DEFINABILITY OF FORCING

By

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Abstract

A new notion of class forcing was defined by M.C. Stanley in [6], which is not *a priori* definable in the ground model. In this thesis, we explore conditions under which the Definability Lemma holds for this new notion.

There are two major approaches to this problem, one direct and one indirect. The direct method makes use of a new complexity class, $\sigma_2 DF_n$, which lies intermediate between Σ_n/Π_n and Δ_{n+1} in the Levy hierarchy. This class has many nice closure properties; in particular, it is "weakly" closed under negation and bounded quantifiers, and is "weakly" self-defining where this notion of "weakness" is inherent in the definition of $\sigma_2 DF_n$. Using this class, one can then show that this new forcing is definable for a wide range of partial orders.

The indirect approach explores the relationship between forcing over an order and forcing over a dense suborder. This is trivial when these orders are sets in the ground model, but the situation is more complicated when both are classes and is in fact false in general. We prove that, even for this new notion of forcing, dense suborders produce the expected results for another wide range of partial orders.

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Chapter 1

Introduction

1.1 Overview

A central question in set theory concerns the power and generality of the forcing method. There are many ways of formalizing this question, one of the most obvious being: if two theories are equiconsistent, can this always be witnessed by a forcing argument? Are there really no limits to the technique beyond the mild restriction that we must always use a (pre)order?

As this question is almost too general to be comprehended, in the early 1970s Solovay restricted the question to a more tractable setting: when is a real realizable in a forcing extension of another? Although it may seem minor to the modern eye, the change of perspective was crucial here, as it turned what had been a purely syntactic question about provability into a semantic notion involving actual models of set theory.

Let us recapitulate Solovay's construction: given two reals $x, y \in \omega^\omega$, one can ask whether one real, in some sense, "forces" the existence of the other. Regarding $\mathbb{L}$, the constructible sets, as being the ZFC closure of a given set or class, we regard x as a "stronger" real than y iff

$$y \in \mathbb{L}[x]$$

It is elementary that this induces a preorder on the reals, whose degree structure has been well-studied, and it is also elementary that $0^\sharp$ cannot be realized in any forcing extension of $\mathbb{L}$. The celebrated "coding the universe in a real" argument of Beller, Jensen and Welch ([1]) actually shows that there are reals strictly weaker than $0^\sharp$ in the constructibility order which cannot be realized by any forcing extension, thus completely ruining any hope of using forcing to somehow surmount the universe.

What has not been well-studied is a more general relationship that relates, not reals, but actual countable transitive models of set theory. That is, instead of considering models of the form $\mathbb{L}[x]$ as above – which, provided we have sufficient strength in the universe, we can – we can more generally consider the collection

$$\mathcal{M}_\delta = \{M : |M| = \omega \wedge M \models \text{ZFC} \wedge o(M) = \delta\}$$

the collection of all ctms with ordinals δ. The question is then, how strong is the forcing method in this collection? Is there a natural ordering on these models akin to the constructibility degrees and, if so, how does it interact with equiconsistency or forcing techniques?

This question is of broader interest than it might at first appear. In some sense the archetypical set theoretic result is a consistency proof: a theory T is consistent with another T'. This is a *syntactic* relation: when all is said and done, one can (in theory) unravel the semantic concepts of countable transitive models to produce a finitistic formal proof of the relative consistency of the two theories. Furthermore, these theories can be given a natural consistency ordering, where $T \equiv T'$ iff $Con(T) \leftrightarrow Con(T')$ and $T < T'$ iff $T' \vdash Con(T)$.

Returning to Solovay's insight, then, a natural question is: can this notion of consistency strength extend to actual models themselves? In other words, can we extend consistency from the syntactic to the semantic?

The obvious extensions are quickly seen to fail; it is possible for a model to, say, not possess $0^\sharp$ but have objects whose consistency strength exceeds it.[1] The problem is that, in some sense, we have failed to ask the right question to gauge our model's strength. Erring on the side of caution, we could instead try to compare the full theories of the models to one another, but this is useless since theories are complete, and hence incomparable. Again, one could try to ferret out particular sentences that represent the model's true strength, but this does not seem to be natural, and certainly fails to be practical.

There is, however, a result by MC Stanley that sheds new light on the subject. In [6] Stanley proved the following:

Theorem 1.1. *Suppose that V is a countable transitive model of $\mathrm{ZFC} + 0^\sharp$ and let L be its constructible inner model. Then there is a V-definable, L-amenable partial order $\mathbb{P}$ and an appropriately generic filter (see below) G such that $V = L[G]$.*

Thus, restricting our attention to

$$\mathcal{Z}_\delta = \{M : |M| = \omega \wedge M \models \mathrm{ZFC} + 0^\sharp \wedge o(M) = \delta\}$$

those models which contain a sharp, we now have a possible way of gauging the consistency strength of models: one model is stronger than another if it can somehow "induce" a generic filter whose ZFC-closure is the other model. This, and matters like it, will be explored later.[2]

[1] Harvey Friedman's work on Boolean Relation Theory is filled with such examples.

[2] Note that we have passed beyond the realm of independence proofs, since the theory $\mathrm{ZFC} + 0^\sharp$ is strictly stronger than that of $\mathrm{ZFC} + V = L$. Forcing has become an end unto itself.

There is a slight catch, however: Stanley's result is, on first blush, impossible, flat-out contradictory to what was stated above. The crucial point is that there are multiple notions of genericity at the class level, all generalizations of the usual notions of set genericity but with distinct properties. While it is true that the result is impossible for the usual notion of class genericity (see [2] for an easy proof), Stanley develops a notion of genericity which he simply calls "generic", but which I will call "truth-generic", for which this result holds.

Unfortunately, where the usual notion of class genericity satisfies the Definability Lemma, this new notion *a priori* does not. This, then, is the central topic of this dissertation: under what conditions is truth-generic forcing definable? Using this definability, when can one extend the Solovay ordering of the reals and speak of the relative consistency strength of models? And what other properties of set- and conventional class-generic forcing can be ported to this new kind of forcing?

1.2 Results

Although full answers remain elusive, several partial results have been obtained, of interest in their own rights. They come in two distinct flavors, one direct and one indirect. First, the indirect, which is of a very combinatorial natural:

Theorem. *Suppose that* $(V; \mathbb{P}, \mathbb{Q}) \models \mathrm{ZFC}$ *and that* $\mathbb{P}$ *is a dense suborder of* $\mathbb{Q}$*. If* $\mathbb{Q}$ *is set-splitting over* $\mathbb{P}$*, then: there is a natural correspondence between truth-generic filters* G *over* $\mathbb{P}$ *and truth-generic filters* H *over* $\mathbb{Q}$*;* $V[G] = V[H]$ *for such filters; and* $\mathbb{P}$*-forcing and* $\mathbb{Q}$*-forcing are identical on sentences of the* $\mathbb{P}$*-forcing language.*

Theorem. *If* $\mathbb{P}$ *is a* V*-amenable partial order, then it has a unique set-completion* $\overline{\mathbb{P}} \sqsubset$

$(V;\mathbb{P})$ *into which* $\mathbb{P}$ *densely embeds.*

This latter implies that truth-generic forcing over $\mathbb{P}$ is definable whenever $\mathbb{P}$ is set-splitting in its own completion. The exact combinatorial details of when this happens may be found in Chapters 2 and 3.

On the other hand, attacking the problem head-on yields fruit of its own:

Theorem. *There is a complexity class* $\sigma_2 DF_n$ *that lies strictly between* Σ_n *and* Δ_{n+1} *which is weakly self-defining.*

Theorem. *If* $\mathbb{P}$ *is an amenable partial order such that* $\mathbb{P}$ *is* κ*-closed for every* $\kappa < \infty$*, then truth-generic forcing over* $\mathbb{P}$ *is definable.*

Theorem. *If* $\mathbb{P}$ *is a set-complete boolean algebra then truth-generic forcing over* $\mathbb{P}$ *is definable.*

There is a third, somewhat trivial, class of orders for which forcing is definable:

Theorem. *If* $\mathbb{P}$ *is set-based for truth-genericity, then truth-generic forcing over* $\mathbb{P}$ *is definable.*

These represent a vast improvement over the existing results, wherein truth-generic forcing over $\mathbb{P}$ is *a priori* not definable at all!

1.3 The basic framework

In order to facilitate the exposition of these results, some definitions and conventions must be established. We work throughout in a $\mathbb{V} \models$ ZFC and whatever additional axioms are required for the relevant objects to exist, e.g. countable transitive models of

$0^\sharp$. Countable transitive models (hereafter abbreviated "ctm") will invariably be denoted U, V and W; their common ordinals, though in reality a countable ordinal δ, will be denoted by ∞. Our models will usually be augmented by predicates, and for this we have the fundamental definition:

Definition 1.2. *A class $R \subset V$ is called V-*amenable *(or just* amenable *if the context is understood) iff $(V; R) \models ZFC$ in the language extended by a predicate for R. Two classes R_1, R_2 are said to be* simultaneously amenable *if $(V; R_1, R_2) \models ZFC$ in the language extended by both predicates, and similarly for additional classes.*

It is a basic observation about amenability that while anything definable from an amenable predicate is itself amenable, two classes R_1 and R_2 which are separately amenable may fail to be simultaneously amenable (cf [2]). Since definability is of paramount importance, we will make the following definitions:

Definition 1.3. *A subset A of a ctm V will be called a* class *of V. Those subsets B which are definable over V – in other words, those subsets which are classes in the usual first-order sense – will be called* definable classes *of V, and the relationship between the two will be denoted $B \sqsubset V$. Likewise, if B is a definable class in a model V augmented by a class A, we will write $B \sqsubset (V; A)$.*

Since classes play double-duty in what will follow, both predicate and object, we will sometimes use the restriction operator, $\restriction$, for more suggestive results. For example, if $\mathbb{P} \subset \mathbb{Q}$ and H is a filter on $\mathbb{Q}$, we will sometimes refer to $H \cap \mathbb{P}$ as $H \restriction \mathbb{P}$. Likewise, a statement like "$\Vdash_{\mathbb{P}} = \Vdash_{\mathbb{Q}} \restriction \mathbb{P}$" translates as the far more cumbersome: for every $p \in \mathbb{P}$ and every $\varphi \in \mathcal{L}^{\mathbb{P}} \subset \mathcal{L}^{\mathbb{Q}}$, $p \Vdash_{\mathbb{P}} \varphi$ iff $p \Vdash_{\mathbb{Q}} \varphi$.

1.4 Partial orders

Now the single most important basic convention and definition.

Definition 1.4. *Throughout,* $\mathbb{P}$ *(and* $\mathbb{Q}$*) will be amenable class partial orders in* V*. This means that (abusing notation slightly) the graph of* $\mathbb{P}$*, considered as a class in* V*, is amenable; alternatively, that* $\leq_{\mathbb{P}}$ *is an amenable class.*

There is a natural topological space associated with every partial order $\mathbb{P}$:

Definition 1.5. *The* Stone Space *of* $\mathbb{P}$*,* $\mathrm{St}(\mathbb{P})$*, is the set of all strongly maximal filters on* $\mathbb{P}$ *– i.e. those maximal filters* $G \subset \mathbb{P}$ *such that* $p \notin G$ *implies there exists a* $q \in G$ *with* $q \perp p$ *– topologized with the open basis* $O_p = \{G \in \mathrm{St}(\mathbb{P}) : G \ni p\}$*.*

Remark 1.6. *Since verifying the* strong *maximality of the filters will never be an issue in this paper, we will simply refer to such filters as* maximal *hereafter.*

The following are similar in notation and obviously related as concepts, but which must be kept straight.

Definition 1.7. *For* $p \in \mathbb{P}$*, define:*

- $p^{\perp} = \{q \in \mathbb{P} : q \perp p\}$
- $[p] = p\downarrow = \{q \in \mathbb{P} : q \leq p\}$*.* $[p]$ *is a subset of* $\mathbb{P}$*.*
- $[\![p]\!] = \{G \in \mathrm{St}(\mathbb{P}) : G \ni p\}$ *(which is just a more tractable notation for the* O_p *defined above).* $[\![p]\!]$ *is a subset of* $\mathrm{St}(\mathbb{P})$*.*

Extend these definitions to sets $A \subset \mathbb{P}$ *in the obvious way:*

- $A^\perp = \{q \in \mathbb{P} : \forall r{\in}A\,(q \perp r)\}$

- $[A] = \{q \in \mathbb{P} : \exists r{\in}A\,(q \leq r)\}$.

- $[\![A]\!] = \{G \in \mathrm{St}(\mathbb{P}) : \exists p{\in}A\,(G \ni p)\}$ *as defined above.*

As usual, we denote the set of (Schoenfield) $\mathbb{P}$-names as $V^{\mathbb{P}}$ instead of the fussier $(V;\mathbb{P})^{\mathbb{P}}$. When $\mathbb{P} \subset \mathbb{Q}$ we consider $V^{\mathbb{P}} \subset V^{\mathbb{Q}}$ in the obvious manner. Elements of $V^{\mathbb{P}}$ will be denoted with circles above the letter, e.g. $\mathring{a}$. Since many of the constructions we will be employing will be recursive over the structure of the name, we make the following definitions:

Definition 1.8. *Fix a model V and an amenable order $\mathbb{P}$. Given a name $\mathring{a}$, we define:*

- *The* domain *of $\mathring{a}$ is the collection of all names appearing in $\mathring{a}$, i.e.*

$$\mathrm{dom}(\mathring{a}) = \{\mathring{b} : \exists p{\in}\mathbb{P}\,((\mathring{b}, p) \in \mathring{a})\}$$

- *The* domain closure *of $\mathring{a}$ is defined recursively as the union of all names appearing in the construction of $\mathring{a}$, i.e.*

$$\mathrm{domcl}(\mathring{a}) = \bigcup\{\mathrm{domcl}(\mathring{b}) : \mathring{b} \in \mathrm{dom}(\mathring{a})\} \cup \mathrm{dom}(\mathring{a})$$

- *The* conditions *of $\mathring{a}$ are the elements of $\mathbb{P}$ appearing in $\mathring{a}$, i.e.*

$$\mathrm{cond}(\mathring{a}) = \{p : \exists \mathring{b}{\in}V^{\mathbb{P}}\,((\mathring{b}, p) \in \mathring{a})\}$$

- *The* support *of $\mathring{a}$ is defined recursively as the union of all conditions of elements of $\mathring{a}$, i.e.*

$$\mathrm{supt}(\mathring{a}) = \bigcup\{\mathrm{supt}(\mathring{b}) : \mathring{b} \in \mathrm{dom}(\mathring{a})\} \cup \mathrm{cond}(\mathring{a})$$

- *Finally, let* $\mathring{a}(\mathring{b}) = \{p \in \mathrm{cond}(\mathring{a}) : (\mathring{b}, p) \in \mathring{a}\}$.

As much of what we will be concerned with are dense suborders, we make the following notational convention: $A \subset^* B$, where both are subsets of a common order $\mathbb{P}$, means that A is a dense subset of B. There is a more general notion of *dense equivalence* that will not be needed here; suffice to say that $A \subset^* B$ in the notation of this paper is a special case of A and B being dense equivalent, i.e. A dense in B and B dense in A.

We now discuss briefly the notion of "genericity" for filters. For the usual notion of set genericity, see either Kunen [4] for the order-theoretic exposition or Jech [3] for the Boolean-algebraic version. The standard text for forcing at the class level is Friedman [2], whose "generic" is what Stanley terms "definably generic" in [6]:

Definition 1.9. *Let* $(V; \mathbb{P}) \models \mathrm{ZFC}$ *with* $\mathbb{P}$ *an amenable partial order. Let* G *be a maximal filter on* $\mathbb{P}$*. Then* G *is said to be* definably generic *if, given any dense class* $D \sqsubset (V; \mathbb{P})$ *(possibly defined with parameters),* $G \cap D \neq \emptyset$*.* G *is said to be* internally generic *if, given any predense set* $E \in V$*,* $G \cap E \neq \emptyset$*.*

There is a third notion of genericity developed in [6] which is the subject of the present paper; its definition and consequences will be given in the next chapter.

1.5 Coding

The last conventions that must be established concern the coding of sentences and formulas in various formal languages. Most important is that while the specifics of the coding of formulas will change occasionally, one rule rises above all: no stupid coding. In particular, the coding of all predicates should be Δ_0 in their parameters, and indeed, considered as a formula, computable in its variables. Beyond that, essentially no details of the actual coding (e.g. sequences of prime factorizations) will, or need, be given.

The basic language in which we shall work is $\mathcal{L} = \{\in, =\}$, the language of set theory, possibly augmented by predicate symbols for amenable classes. Things become trickier when dealing with the forcing languages. There, our basic language will consist of two symbols $\widehat{\in}$ and $\widehat{\subset}$, to be interpreted in the obvious ways;[3] these predicate symbols are then augmented by constant symbols for every element of $V^{\mathbb{P}}$ to form the *forcing language* of $\mathbb{P}$, denoted by $\mathcal{L}^{\mathbb{P}}$, and may or may not contain a symbol for $\mathbb{P}$ itself depending on context. As before, if $\mathbb{P} \subset \mathbb{Q}$, we regard $\mathcal{L}^{\mathbb{P}}$ sentences as $\mathcal{L}^{\mathbb{Q}}$ sentences as well; and indeed, for notational convenience, we will simply write $\mathcal{L}^{\mathbb{P}}$ for the collection of $\mathcal{L}^{\mathbb{P}}$ sentences, so this may be written more compactly as $\mathcal{L}^{\mathbb{P}} \subset \mathcal{L}^{\mathbb{Q}}$.

Finally, the strict adherence to the hat over $\widehat{\in}$ and $\widehat{\subset}$, and the circle over $\mathring{a}$, may seem unnecessarily fussy, but it is enormously helpful when dealing with equations involving the F function defined in the next chapter.

[3] As the cognoscenti will doubtless recognize, forcing for partial orders is more naturally defined with the subset relation than with equality, which is a somewhat irritating extra predicate.

Chapter 2

Combinatorics of Truth-Genericity

The basic mechanics of set-forcing are well-understand and rather easy to implement: one decides upon a property to be witnessed in the extension model and then cooks up dense sets or maximal antichains that will force this property to be realized. These techniques port very well to the definably-generic case, as one might expect, since the basic property of genericity – intersecting dense things one can define in a more or less arbitrary fashion – remains the same. For example, the following is a trivial exercise with definable genericity:

Lemma 2.1. *Suppose G is definably generic and $q \notin G$. Then there is an $r \in G$ such that $q \perp r$.*

The key is that the set $D = \{q\} \cup q^{\perp}$ is both dense and definable. When working with truth-genericity, however, the argument fails: we are guaranteed the intersection of *some* dense classes, but only those of a specific type, namely $F(\varphi)$ for some φ in the forcing language. As such, there are a wealth of combinatorial results about truth-generic filters that must be re-established using new techniques that translate our desired property into, not dense sets, but sentences of the forcing language.

2.1 Definitions

Herewith is the definition of forcing as found in Stanley [6], used as the precursor to defining syntactic forcing:

Definition 2.2. *For any partial order $\mathbb{P}$, we define, by hyperrecursion, the following subclasses of $\mathbb{P}$:*

$$\begin{aligned}
F^+(\mathring{a}\,\widehat{\in}\,\mathring{b}) &:= \bigcup_{\mathring{d}\in\mathrm{dom}(\mathring{b})} \Big([\mathring{b}(\mathring{d})] \cap F^+(\mathring{a}\widehat{\subset}\mathring{d}) \cap F^+(\mathring{d}\widehat{\subset}\mathring{a})\Big) \\
F^-(\mathring{a}\,\widehat{\in}\,\mathring{b}) &:= F^+(\mathring{a}\widehat{\in}\mathring{b})^\perp \\
F^-(\mathring{a}\,\widehat{\subset}\,\mathring{b}) &:= \bigcup_{\mathring{c}\in\mathrm{dom}(\mathring{a})} \Big([\mathring{a}(\mathring{c})] \cap F^-(\mathring{c}\widehat{\in}\mathring{b})\Big) \\
F^+(\mathring{a}\,\widehat{\subset}\,\mathring{b}) &:= F^-(\mathring{a}\widehat{\subset}\mathring{b})^\perp \\
F^+(\varphi \wedge \psi) &= F^+(\varphi) \cap F^+(\psi) \\
F^+(\exists x \varphi(x)) &= \bigcup_{\mathring{a}\in V^{\mathbb{P}}} F^+(\varphi(\mathring{a}/x))
\end{aligned}$$

If $F^{\pm}(\varphi)$ has already been defined by the above, let $F^{\mp}(\varphi) = [F^{\pm}(\varphi)]^\perp$.

We then say

$$p \Vdash \varphi \qquad \textit{iff} \qquad F^+(\varphi) \textit{ is dense below } p$$

and that G is truth-generic *if*

$$V[G] \models \varphi \Leftrightarrow \exists p{\in}G\,(p \Vdash \varphi)$$

Remark 2.3. *Note that the definitions of $F^-(\mathring{a}\widehat{\in}\mathring{b})$ and $F^+(\mathring{a}\widehat{\subset}\mathring{b})$ are redundant given the definition of negation. We include them only to make the arguments in Chapter 4 more readable.*

There is an equivalent notion, termed "semantic genericity" by Stanley. We repeat the definition from Chapter 1 for completeness' sake:

Definition 2.4. *The* Stone Space *of* $\mathbb{P}$ *is the collection of all maximal filters on* $\mathbb{P}$ *topologized by the open basis* $[\![p]\!] = \{G : G \ni p\}$*. Let* $\mathrm{Mod}(\varphi) = \{G : V[G] \models \varphi\}$*; then* $p \mid\!\models \varphi$ *iff* $\mathrm{Mod}(\varphi) \cap [\![p]\!]$ *is comeager in* $[\![p]\!]$*. Once again, we say that* G *is (semantically) truth-generic if it satisfies* $V[G] \models \varphi \Leftrightarrow \exists p{\in}G\, (p \mid\!\models \varphi)$*.*

A straightforward recursion argument shows that syntactic and semantic forcing are identical, so we shall denote both by $\Vdash$. Although we will primarily be concerned with the former, the latter will sometimes be employed for ease of use; and provided one is careful, this will not impact the definability of the relevant properties.

2.2 Set-based Orders

The following is an incredibly useful example – or, perhaps, counter-example – illustrating the distinction between forcing at the set level and forcing at the class level. Given a uniform sequence of partial orders $\langle \mathbb{P}_\alpha : \alpha < \infty \rangle$, define the so-called *lottery preparation* of the orders to be $\mathbb{P} = \bigsqcup \{\alpha\} \times \mathbb{P}_\alpha$ with a unique maximal element dominating each of the $\mathbb{P}_\alpha$. The *slanted lottery preparation* is given by the same underlying class, but ordered so that $\mathbf{1}_\alpha \leq \mathbf{1}_\beta$ whenever $\beta < \alpha$, where $\mathbf{1}_i$ is the maximal element of $\mathbb{P}_i$.[1] For our purposes, it will suffice to consider $\mathbb{P}_\alpha = \mathrm{Fn}(\omega_{\alpha+1}, 2)$, which we will call the Cohen lottery and slanted Cohen lottery preparations respectively.

[1]One could obviously adapt this for orders that do not have a maximal element but it eases the exposition to assume that they do – and, indeed, the Cohen orders have them natively.

Lemma 2.5. *Letting $\mathbb{P}$ be the Cohen lottery (resp. the slanted Cohen lottery preparation), $\mathbb{P}$ is neither κ-complete for every κ, nor is set-splitting in its own completion (cf Chapter 3).*

Proof. The first is obvious. For the second, pick any $p \in \mathbb{P}_\alpha$ and look at $\neg p \in \overline{\mathbb{P}}$; this clearly bounds every $q \in \mathbb{P}_\beta$ where $\beta \neq \alpha$, and hence does not set-split over $\mathbb{P}$. □

So $\mathbb{P}$ defies the techniques that will be laid forth in the following chapters. This should not be worrying, though, as it seems as if $\mathbb{P}$ "ought" to be behave just like ordinary Cohen forcing, and hence ought to be definable; and indeed it is.

Theorem 2.6. *Forcing over $\mathbb{P}$ is definable.*

The proof of this is slightly more complicated than it might appear at first blush.

Proposition 2.7. *The following hold for the Cohen Lottery $\mathbb{P}$:*

1. *Any (maximal) filter on $\mathbb{P}$ restricts to a unique (maximal) filter on a single $\mathbb{P}_\alpha$; conversely, any (maximal) filter on $\mathbb{P}_\alpha$ extends to a unique (maximal) filter on $\mathbb{P}$. Furthermore, $\mathrm{St}(\mathbb{P}) \approx \bigsqcup \mathrm{St}(\mathbb{P}_\alpha)$.*

2. *Generic filters on $\mathbb{P}$ restrict to generic filters on $\mathbb{P}_\alpha$, while generic filters on $\mathbb{P}_\alpha$ extend to generic filters on $\mathbb{P}$, for either truth or definable genericity.*

3. *$p \Vdash_{\mathbb{P}} \varphi$ iff $p \Vdash_{\mathbb{P}_\alpha} \varphi'$, where φ' is the result of hereditarily deleting all conditions in φ (i.e. all conditions in the support of every name in φ) that are incompatible with $\mathbb{P}_\alpha$.*

4. *$p \Vdash_{\mathbb{P}_\alpha} \varphi$ is uniformly definable in p and α for fixed φ, where we again delete conditions in φ that are incompatible with $\mathbb{P}_\alpha$.*

Hence forcing is definable for $\mathbb{P}$.

Proof. Although this may be proven directly from the combinatorics of F, it is much easier to do so using semantic forcing. Let $\mathbf{1}_\alpha$ denote the largest element of $\mathbb{P}_\alpha$; then the neighborhoods $[\![\mathbf{1}_\alpha]\!] \subset \mathrm{St}(\mathbb{P})$ are disjoint. This means that if $p \in \mathbb{P}_\alpha$, then any comeager neighborhood of $[\![p]\!]$ in $\mathrm{St}(\mathbb{P})$ is necessarily a comeager neighborhood of $[\![p]\!]$ in $\mathrm{St}(\mathbb{P}_\alpha)$ and vice versa, so $p \Vdash_{\mathbb{P}} \varphi$ iff $p \Vdash_{\mathbb{P}_\alpha} \tilde{\varphi}$, where we can delete "inappropriate" elements of φ again by comeagerness. This deletion can be defined uniformly in α and so the forcings can be defined uniformly in α also. Hence G is generic over $\mathbb{P}$ iff G is generic over $\mathbb{P}_\alpha$.

□

The essential point here is that forcing over $\mathbb{P}$ is somehow "set-based": there is a uniform sequence of $\mathbb{P}_\alpha \subset \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ reduces to $p \Vdash_{\mathbb{P}_\alpha}$, where the latter relation is a) set-forcing and b) uniformly definable. As such, we have the following notion:

Definition 2.8. *A partial order $\mathbb{P}$ is* strongly set-based *if there is a uniform collection of orders $\mathbb{P}_\alpha \subset \mathbb{P}$ such that*

1. *Each $\mathbb{P}_\alpha$ is a set, i.e. $\mathbb{P}_\alpha \in V$*
2. *Every maximal chain on $\mathbb{P}$ intersects at least one $\mathbb{P}_\alpha$.*

Remark 2.9. *The "strongly" here denotes that every maximal filter intersects one of these $\mathbb{P}_\alpha$. See Theorem 2.15.*

We can now generalize the previous theorem.

Theorem 2.10. *If $\mathbb{P}$ is strongly set-based, then $\Vdash_{\mathbb{P}}$ is definable.*

This is not quite as trivial as it might first seem; essentially, we need to confirm that $\Vdash_{\mathbb{P}_\alpha} = \Vdash_\alpha$ is the same as $\Vdash_{\mathbb{P}}$. To that end, we will prove several propositions first.

Proposition 2.11. *If $\mathbb{Q} \subset \mathbb{P}$ is open, there is a canonical homeomorphism $\pi : \mathrm{St}(\mathbb{Q}) \approx [\![\mathbb{Q}]\!]_{\mathbb{P}}$ given by $H \mapsto H \uparrow_{\mathbb{P}}$, and whose inverse is given by $G \mapsto G \cap \mathbb{Q}$.*

Proof. To begin, let us show that this map is well-defined. Note that if G is a filter on $\mathbb{P}$ then $G \cap \mathbb{Q}$ is a filter on $\mathbb{Q}$ because $\mathbb{Q}$ is open. Now suppose that H is a maximal filter on $\mathbb{Q}$ and let G be its upward closure in $\mathbb{P}$. If G is not maximal, then there is a $G' \supset G$. Clearly $G' \cap \mathbb{Q}$ is a filter on $\mathbb{Q}$ which contains H, and since H is maximal we must have $G' \cap \mathbb{Q} = H$. Hence, if $G' \neq G$, there is some $p \in G' \setminus G$ and therefore a $q \in G' \cap \mathbb{Q}$ with $q \leq p$; but then $p \in G$, which is a contradiction. Conversely, let H be a maximal filter on $\mathbb{Q}$ containing $G \cap \mathbb{Q}$. By the above, there is a unique maximal filter on $\mathbb{P}$ containing H, but G is a maximal filter on $\mathbb{P}$ containing H, so π is actually a bijection.

Finally, to prove it is a homeomorphism, it suffices to establish that π is an open, continuous map. Let $p \in \mathbb{P}$ be given; then (abusing notation slightly) $\pi^{-1}[\![p]\!]_{\mathbb{P}} = \bigcup\{[\![q]\!]_{\mathbb{Q}} : q \in \mathbb{Q} \wedge q \leq p\}$, so π is continuous, while π is open because $\pi[\![q]\!]_{\mathbb{Q}} = [\![q]\!]_{\mathbb{P}}$. □

As an aside, the reader may recall a similar theorem from boolean algebra theory, which states that if $\mathbb{P}, \mathbb{Q}$ are both boolean algebras and $\mathbb{Q} \subset^* \mathbb{P}$, then $\mathrm{St}(\mathbb{Q}) \approx \mathrm{St}(\mathbb{P})$. This statement does *not* hold for arbitrary partial orders, however. Consider the following example, which is extraordinarily simple if one draws a picture:

- Let $\mathbb{P}_0$ and $\mathbb{P}_1$ be copies of $2^{<\omega}$
- Let σ_0 be the rightmost branch of $\mathbb{P}_0$ (i.e. σ_0 consists of all sequences of only 0s in $\mathbb{P}_0$) and σ_1 be the leftmost branch of $\mathbb{P}_1$ (i.e. σ_1 consists of all sequences of only 1s in $\mathbb{P}_1$); and let $\mathbb{Q}_i = \mathbb{P}_i \setminus \sigma_i$.

- Glue the orders together starting at the 17th places of σ_0 and σ_1 respectively; that is, let $\mathbb{P} = \mathbb{P}_0 \cup \mathbb{P}_1$ and regard $(0^n)_{\mathbb{P}_0} \equiv (1^n)_{\mathbb{P}_1}$.

- Let $\mathbb{Q} = \mathbb{Q}_0 \cup \mathbb{Q}_1 \cup \{00000_{\mathbb{P}_0}\} \cup \{11111_{\mathbb{P}_1}\}$

- Let $G = \sigma_0 \cup \sigma_1$.

Then G is a maximal filter on $\mathbb{P}$, $\mathbb{Q}$ is dense in $\mathbb{P}$, but $G \cap \mathbb{Q} = \{00000_{\mathbb{P}_0}\} \cup \{11111_{\mathbb{P}_1}\}$ isn't actually filter on $\mathbb{Q}$ since they have no common extension in $\mathbb{Q}$. The best one can say is that $G \cap \mathbb{Q}$ is "maximal" in the sense that no filter on $\mathbb{Q}$ properly contains it. The reason for this discrepancy is that when $\mathbb{P}$ and $\mathbb{Q}$ are boolean algebras, any filter on one order is guaranteed to have the same witnesses to its "filterdom" on the other (namely their mutual meet); it fails to work here because $\mathbb{Q}$ may be arbitrarily complicated inside $\mathbb{P}$ while still remaining dense.

Returning to the topic at hand, the next proposition, though trivial, is ubiquitous:

Proposition 2.12 (The Renaming Lemma)**.** *Suppose $\mathbb{Q} \subset \mathbb{P}$ and let $H = G \cap \mathbb{Q}$. Suppose further that there is a map $\eta : V^{\mathbb{P}} \to V^{\mathbb{Q}}$ such that, for every $\mathring{a} \in V^{\mathbb{P}}$, we have*

$$\mathring{a}_G = (\eta(\mathring{a}))_G = (\eta(\mathring{a}))_H$$

Then:

1. $V[G] = V[H]$

2. *If we let $\tilde{\varphi} = \eta(\varphi)$, i.e. the result of replacing all $\mathring{a} \in \varphi$ by $\eta(\mathring{a})$, then*

$$V[G] \models \varphi \Leftrightarrow V[H] \models \tilde{\varphi}$$

3. $\pi[\mathrm{Mod}_{\mathbb{P}}(\varphi)] = \mathrm{Mod}_{\mathbb{Q}}(\tilde{\varphi})$

Proof. $V[H] \subset V[G]$ automatically, and η guarantees that $V[G] \subset V[H]$. The latter parts follow from the definition of the forcing language and induction on the complexity of φ. $\square$

Naturally, if $\mathbb{P}$ is set-based, we can produce such a map η.

Lemma 2.13. *Suppose $\mathbb{Q} \subset \mathbb{P}$ is both open and a set and G is a maximal filter that meets $\mathbb{Q}$. Letting $H = G \cap \mathbb{Q}$, one can uniformly define an η as in the previous proposition.*

Proof. This is a special case of the Regularization Lemma so we will somewhat abbreviate the proof. Given a name $\mathring{a} \in V^{\mathbb{P}}$, we will recursively map every condition $p \in \mathbb{P}$ to $e_p^+ = \{q \in \mathbb{Q} : q \leq p\}$. Since $G \cap \mathbb{Q} \neq \emptyset$, it follows that $G \cap e_p \neq \emptyset$ iff $p \in G$, where $e_p = \{q \in \mathbb{Q} : q \leq p \wedge q \perp p\}$ is dense in $\mathbb{Q}$. Hence, letting $\mathring{b}$ be the result of this renaming, $\mathring{a}_G = \mathring{b}_G = \mathring{b}_H$ as required. The definition is uniform because the map $p \mapsto e_p^+$ is uniform in p and $\mathbb{Q}$. $\square$

The reader who is unsatisfied by the above proof is encouraged to read ahead to Lemma 2.37, where all the details are accounted for.

Theorem 2.14. *Suppose that $\mathbb{Q} \subset \mathbb{P}$ is both open and a set. Then, for every $q \in \mathbb{Q}$ and for every sentence $\varphi \in \mathcal{L}^{\mathbb{P}}$, we have*

$$q \Vdash_{\mathbb{P}} \varphi \quad \textit{iff} \quad q \Vdash_{\mathbb{Q}} \tilde{\varphi}$$

where $\tilde{\varphi}$ is defined as above.

Proof. This follows immediately from the previous results. Since $q \in \mathbb{Q}$ and $\mathbb{Q}$ is an open subset of $\mathbb{P}$, it follows that $\pi : [\![q]\!]_{\mathbb{P}} \approx [\![q]\!]_{\mathbb{Q}}$ and hence $\mathrm{Mod}_{\mathbb{P}}(\varphi) \cap [\![\varphi]\!]_{\mathbb{P}}$ is homeomorphic to $\mathrm{Mod}_{\mathbb{Q}}(\tilde{\varphi}) \cap [\![q]\!]_{\mathbb{Q}}$. Thus the sets computed in $\mathbb{P}$ are comeager iff the sets computed in $\mathbb{Q}$ are comeager, proving the theorem. $\square$

Proof of Theorem 2.10. Since $\Vdash_\alpha$ is uniformly definable, and since every maximal filter intersects some $\mathbb{P}_\alpha$, by the previous theorem we can define

$$p \Vdash \varphi \qquad \text{iff} \qquad \{q \in \mathbb{P} : q \in \mathbb{P}_\alpha \wedge q \Vdash_\alpha \tilde{\varphi}_\alpha\} \text{ is dense below } p$$

is dense below p, where we let $\tilde{\varphi}_\alpha$ denote the regularization of φ into $\mathbb{P}_\alpha$. □

Note that we can generalize the theorem still further, using an identical proof:

Theorem 2.15. *Say that a partial order $\mathbb{P}$ is* set-based *for a type of genericity $\mathcal{G}$ if there is a uniform collection of orders $\mathbb{P}_\alpha \subset \mathbb{P}$ such that*

1. *Each $\mathbb{P}_\alpha$ is a set, i.e. $\mathbb{P}_\alpha \in V$*

2. *Every $\mathcal{G}$-generic filter intersects at least one $\mathbb{P}_\alpha$.*

Then $\mathcal{G}$-forcing is definable over orders that are set-based for $\mathcal{G}$.

Hence, for example, if $A \subset \mathbb{P}$ is a definable maximal antichain such that $[a] \in V$ for each $a \in A$, then definably-generic forcing over $\mathbb{P}$ is definable.

The lottery preparations illustrate even more strangeness that can occur at the class level:

Theorem 2.16. *Let $\mathbb{P}$ be the Cohen lottery and $\mathbb{P}^s$ the slanted Cohen lottery. Then:*

- *There is a natural correspondence between predense sets in $\mathbb{P}^s$ and collections of predense sets on the $\mathbb{P}_\alpha$, but there are no predense sets at all in $\mathbb{P}$. Therefore, there is a natural correspondence between internally generic filters on $\mathbb{P}^s$ and collections of internally generic filters on $\mathbb{P}_\alpha$, but* every *filter is internally generic filter on $\mathbb{P}$.*

- *The maximum principle fails for $\mathbb{P}$, for any notion of genericity.*

Proof. The first is obvious, so we only need to check the latter. Let $R = \mathcal{P}(\omega)^V$ and consider the following sentence:

$$\varphi(x) \equiv x \subset \omega \wedge x \notin \check{R}$$

This asserts the existence of a new real in the extension model. Trivially, $\mathbf{1} \Vdash \exists x \varphi(x)$ and by the regular maximal principle in V it is clear that, for any $p \in \mathbb{P}_\alpha$, there is a $\mathring{a}_p$ such that $p \Vdash \varphi(\mathring{a}_p)$. However, given any particular name $\mathring{a}$, it is not the case that $\mathbf{1} \Vdash \mathring{a} \neq \emptyset$ (let alone that it is a new real) since any such name is bounded which means that they will evaluate to the empty set under "most" maximal filters. This can be formalized as need be for any of the three notions of genericity as required.

□

2.3 Regularity

We now move back to the general context of order theory and forcing.

Definition 2.17. *If $X \subset \mathbb{P}$ then we say that X is* open *iff it is closed downwards, i.e. if $\forall p{\in}X\, \forall q[q \leq p \to q \in X]$. We say that it is* regular *iff it is closed under "dense below", i.e. whenever X is dense below p then $p \in X$.*

Remark 2.18. *Regular open is also equivalent to the property: $p \notin X$ implies there is a $q \leq p$ with $q \perp X$.*

A few basic facts about regularity are in order, whose proofs are standard:

Lemma 2.19. *1. If $\mathbb{P}$ is separative then $[p]$ is regular for any $p \in \mathbb{P}$*

2. If X_i are regular, so is $\bigcap X_i$

3. *Given any class* $X \subset \mathbb{P}$*, regular or not,* $X^\perp$ *is regular open.*

4. *If* X *is regular open, then* $(X^\perp)^\perp = X$*.*

5. *Given any class* $X \subset \mathbb{P}$*, there is a least regular class* $\mathrm{reg}(X) \supset X$ *given by* $\mathrm{reg}(X) = (X^\perp)^\perp$*.*

6. *If* $\mathbb{P}$ *happens to be a sufficiently complete boolean algebra, then* $\mathrm{reg}(X) = [\bigvee X]$

7. *Regularity is absolute:* $(V;\mathbb{P}) \models$ *"*X *is regular in* $\mathbb{P}$*"* $\Leftrightarrow \mathbb{V} \models$ *"*X *is regular in* $\mathbb{P}$*"*

We recapitulate a definition from the introduction for completeness' sake:

Definition 2.20. *If* $X, Y \subset \mathbb{P}$ *then we say* $X \subset^* Y$ *iff* $X \subset Y$ *and* X *is* dense *in* Y*; that is,* $\forall y{\in}Y\ \exists x{\in}X\ [x \leq y]$.

Hence our earlier definition of forcing may be rewritten as:

Definition 2.21. $p \Vdash \varphi$ *iff* $F^+(\varphi) \subset^* [p]$

We now prove a few simple lemmas on density, most of which will be used in Chapter 4.

Lemma 2.22.

1. *If* $A \subset^* B \subset^* C$ *then* $A \subset^* C$*.*

2. *If* $A_i \subset^* B_i$ *then* $\bigcup A_i \subset^* \bigcup B_i$*. Note that this applies even if* $I = \infty$*.*

3. *If* $A_i \subset^* B_i$*,* $i \in I$*, and* $\mathbb{P}$ *is* $|I|$*-closed, then* $\bigcap A_i \subset^* \bigcap B_i$*.*

4. $A \subset^* \mathrm{reg}(A)$ *for any* A*.*

And a few simple lemmas on the perp and density:

Lemma 2.23. *1. $A \subset B$ implies $B^\perp \subset A^\perp$.*

2. If $A \subset^* B$, then $A^\perp = B^\perp$.

3. If $A \subset^* B$, then $A^{\perp\perp} = B^{\perp\perp} = \operatorname{reg}(A) = \operatorname{reg}(B)$.

4. $(\bigcup A_i)^\perp = \bigcap A_i^\perp$

5. $(\bigcap A_i)^\perp = \operatorname{reg}(\bigcup A_i^\perp)$

Hence

Corollary 2.24. *For any sentence $\varphi \in \mathcal{L}^{\mathbb{P}}$,*

$$\{p : p \Vdash \varphi\} = \operatorname{reg}(F^+(\varphi)) = F^+(\neg\neg\varphi)$$

Here is now the first important lemma concerning density and regularity:

Lemma 2.25. *Suppose, in any of the quantifier-free clauses of the definition of F, we replace a class of the form $F^\pm(\varphi)$ with another class $X \subset^* F^\pm(\varphi)$. Then the resultant class $G(\varphi) \subset^* F(\varphi)$ for all quantifier-free sentences. If, in addition, quantification and negation applied to quantified formulas remains unchanged, then $G(\varphi) \subset^* F(\varphi)$ for all sentences.*

A quick word on what this means is in order: say, for example, in the clause for membership

$$F^+(\mathring{a} \mathrel{\widehat{\in}} \mathring{b}) := \bigcup_{\mathring{d} \in \operatorname{dom}(\mathring{b})} \Big([\mathring{b}(\mathring{d})] \cap F^+(\mathring{a} \mathrel{\widehat{\subset}} \mathring{d}) \cap F^+(\mathring{d} \mathrel{\widehat{\subset}} \mathring{a}) \Big)$$

we replace (for a given $\mathring{a}, \mathring{d}$) the class $F^+(\mathring{a} \mathrel{\widehat{\subset}} \mathring{d})$ by a class $X \subset^* F^+(\mathring{a} \mathrel{\widehat{\subset}} \mathring{d})$. Then this new class, which we will (temporarily) denote $G^+(\mathring{a} \mathrel{\widehat{\in}} \mathring{b}) \subset^* F^+(\mathring{a} \mathrel{\widehat{\in}} \mathring{b})$; and furthermore, we may replace any number of classes in this way provided the replacement is done uniformly. This will give us tremendous flexibility when defining forcing in the next chapters.

Proof. For quantifier-free formulas, note that $\subset^*$ is preserved under all the clauses of the definition because there are at most finitely many intersections, and any non-trivial partial order $\mathbb{P}$ is closed under finite descending chains. The quantification step follows immediately from the fact that density is preserved under ∞-unions (Lemma 2.22) and the fact that $A \subset^* B$ implies $A^\perp = B^\perp$. □

2.4 Basic facts about truth-genericity

While it is natural, in definably-generic forcing, to deal with the forcing relation, the definition of truth-generic forcing suggests that we look at F. Note the following, though:

Lemma 2.26. *$q \Vdash \varphi$ iff $q \in F^+(\neg\neg\varphi)$.*

Proof. $q \Vdash \varphi$ iff $F^+(\varphi)$ is dense below q iff q is in the regular closure of $F^+(\varphi)$ iff $q \in [F^+(\varphi)]^{\perp\perp}$ iff $q \in F^+(\neg\neg\varphi)$. □

Looking directly at F, now, one of the most basic questions is when a set is forced to be empty. To that end, we should look at the relationship between F and names relative to $\emptyset$.

Lemma 2.27. *The following equalities hold:*

1. $F^+(\mathring{c}\widehat{\in}\emptyset) = \emptyset$
2. $F^-(\mathring{c}\widehat{\in}\emptyset) = \mathbb{P}$

3. $F^-(\emptyset \widehat{\subset} \mathring{c}) = \emptyset$

4. $F^-(\mathring{c} \widehat{\subset} \emptyset) = [\mathrm{cond}(\mathring{c})]$

5. $F^+(\mathring{c} \widehat{=} \emptyset) = [\mathrm{cond}(\mathring{c})]^\perp$

Proof. This is a simple matter of following the definitions. For the first, the definition of $F^+_\in$ requires quantifying over conditions in the second coordinate, which is vacuous here. The second is obtained by applying $\perp$ to the first. The third is similar to the first as the quantifier is vacuous. The fourth is marginally trickier:

$$\begin{aligned} F^-(\mathring{c} \widehat{\in} \emptyset) &= \bigcup_{\mathring{d} \in \mathrm{dom}(\mathring{c})} [\mathring{c}(\mathring{d})] \cap F^-(\mathring{d} \widehat{\in} \emptyset) \\ &= \bigcup_{\mathring{d} \in \mathrm{dom}(\mathring{c})} [\mathring{c}(\mathring{d})] \cap \mathbb{P} \\ &= \bigcup_{\mathring{d} \in \mathrm{dom}(\mathring{c})} [\mathring{c}(\mathring{d})] \\ &= [\mathrm{cond}(\mathring{c})] \end{aligned}$$

The final one requires following the observation:

$$\begin{aligned} F^+(\mathring{c} \widehat{=} \emptyset) &= F^+(\mathring{c} \widehat{\subset} \emptyset) \cap F^+(\emptyset \widehat{\subset} \mathring{c}) \\ &= [F^-(\mathring{c} \widehat{\subset} \emptyset)]^\perp \cap \mathbb{P} \\ &= [\mathrm{cond}(\mathring{c})]^\perp \end{aligned}$$

□

Corollary 2.28. *$q \Vdash \mathring{c} = \emptyset$ iff $\mathrm{cond}(\mathring{c})^\perp$ is dense below q iff $q \perp \mathrm{cond}(\mathring{c})$.*

We also have the following facts:

Lemma 2.29. *1. $F^+(\mathring{a} \widehat{\in} \mathring{b}) \supset [\mathring{b}(\mathring{a})]$*

2. *If* $\mathring{a} \subset \mathring{b}$ *then* $F^+(\mathring{a}\widehat{\subset}\mathring{b}) = \mathbb{P}$.

3. *If* $p \in F^+(\mathring{a}\widehat{\in}\mathring{b})$ *then there is a* $\mathring{d} \in \mathrm{dom}(\mathring{b})$ *such that* $p \in F^+(\mathring{a}\widehat{=}\mathring{d})$

Proof. We prove the first two by a double induction. Assuming the second part is true, we have

$$\begin{aligned} F^+(\mathring{a} \in \mathring{b}) &= \bigcup_{\mathring{c}\in\mathrm{dom}(\mathring{b})} [\mathring{b}(\mathring{c})] \cap F^+(\mathring{a}\widehat{\subset}\mathring{c}) \cap F^+(\mathring{c}\widehat{\subset}\mathring{a}) \\ &\supset [\mathring{b}(\mathring{a})] \cap F^+(\mathring{a}\widehat{\subset}\mathring{a}) \\ &= [\mathring{b}(\mathring{a})] \end{aligned}$$

where we recall that $[\mathring{b}(\mathring{a})] = \emptyset$ if $\mathring{a} \notin \mathrm{dom}(\mathring{b})$.

Conversely, assuming the first part is true, we have

$$F^-(\mathring{a}\widehat{\subset}\mathring{b}) = \bigcup_{\mathring{c}\in\mathrm{dom}(\mathring{a})} [\mathring{a}(\mathring{c})] \cap F^-(\mathring{c}\widehat{\in}\mathring{b})$$

Now, if $\mathring{a} \subset \mathring{b}$, then $\mathring{c} \in \mathrm{dom}(\mathring{b})$ too, $[\mathring{a}(\mathring{c})] \subset [\mathring{b}(\mathring{c})]$; and $F^+(\mathring{c} \in \mathring{b}) \supset [\mathring{b}(\mathring{c})]$, so $F^-(\mathring{c} \in \mathring{b}) \subset [\mathring{b}(\mathring{c})]^\perp$. This means that

$$\begin{aligned} [\mathring{a}(\mathring{c})] \cap F^-(\mathring{c}\widehat{\in}\mathring{b}) &\subset [\mathring{b}(\mathring{c})] \cap [\mathring{b}(\mathring{c})]^\perp \\ &= \emptyset \end{aligned}$$

as required.

The third clause simply restates the definition of the membership relation.

□

The following is included as a useful, though trivial, observation:

Proposition 2.30. *If $F^+(\varphi)$ is regular, then $p \Vdash \varphi$ iff $p \in F^+(\varphi)$. In particular, for any two names $\mathring{a}, \mathring{b}$, $p \Vdash \mathring{a} \subset \mathring{b}$ iff $p \in F^+(\mathring{a} \widehat{\subset} \mathring{b})$.*

Which in turns leads to the following, more useful for its technique than its substance:

Proposition 2.31. *Suppose that φ, ψ are sentences in the forcing language such that*

1. *$F^+(\psi)$ is regular.*
2. *$\vdash \varphi \rightarrow \psi$, where provability is with respect to a) classical logic over b) some theory T such that $\mathbf{1} \Vdash T$.*

Then $F^+(\varphi) \subset F^+(\psi)$.

Proof. We will again use semantic forcing to establish this result. By the previous proposition $p \in F^+(\psi)$ iff $p \Vdash \psi$ and we may assume without loss of generality that the same holds for φ. Now suppose G is a truth-generic filter containing p. Since $p \Vdash \varphi$, we must have $V[G] \models \varphi$, whence $V[G] \models \psi$ because this is a ZFC-provable consequence of φ. Since generic filters are comeager, we conclude that p semantically forces ψ, whence $p \Vdash \psi$, whence $p \in F^+(\psi)$ as required. $\square$

Corollary 2.32. *Let $\mathring{a}, \mathring{b}, \mathring{c} \in V^{\mathbb{P}}$. Then $F^+(\mathring{a} \widehat{\subset} \mathring{b}) \cap F^+(\mathring{b} \widehat{\subset} \mathring{c}) \subset F^+(\mathring{a} \widehat{\subset} \mathring{c})$. Also, $F^+(\mathring{a} \widehat{=} \mathring{b}) \cap F^+(\mathring{b} \widehat{=} \mathring{c}) \subset F^+(\mathring{a} \widehat{=} \mathring{c})$.*

Proof. One can either prove this directly or via this previous proposition, since all the sets in question are regular. $\square$

For the next result, we will need a number of very small lemmas as preamble.

Lemma 2.33. *For any filter G:*

1. $\check{x}_G = x$ *for any* $x \in V$.

2. *If* $a \in b$ *then* $\check{a}_G \in \check{b}_G$ *and* $V[G] \models \check{a} \in \check{b}$.

3. *If* $a \notin b$ *then* $\check{a}_G \notin \check{b}_G$ *and* $V[G] \models \check{a} \notin \check{b}$.

Lemma 2.34. *For any partial order* $\mathbb{P}$ *and* $a, b \in V$*:*

1. *If* $a \in b$ *then* $F^+(\check{a}\widehat{\in}\check{b}) = \mathbb{P}$ *and hence* $F^-(\check{a}\widehat{\in}\check{b}) = \emptyset$.

2. *If* $a \notin b$ *then* $F^-(\check{a}\widehat{\in}\check{b}) = \mathbb{P}$ *and hence* $F^+(\check{a}\widehat{\in}\check{b}) = \emptyset$.

Proposition 2.35. *Suppose* $e \subset \mathbb{P}$ *is a set, i.e.* $e \in V$*, that is predense below some* $p \in \mathbb{P}$*, and that* G *is a truth-generic filter containing* p*. Then* $G \cap e \neq \emptyset$.

Proof. Consider the $\mathbb{P}$-name $\mathring{b} = \{(\emptyset, r_i) : r_i \in e\}$. $G \cap e \neq \emptyset$ is thus equivalent to showing that $V[G] \models \mathring{b} \neq \emptyset$ or, alternatively, $V[G] \models \mathring{b} \not\subset \emptyset$. By truth-genericity, this is equivalent to showing

$$G \cap F^+(\mathring{b}\widehat{\subset}\emptyset) = \emptyset$$

Let us consider $F^-(\mathring{b}\widehat{\subset}\emptyset)$. By the inductive definition, we have

$$\begin{aligned} q \in F^-(\mathring{b}\widehat{\subset}\emptyset) \quad &\Leftrightarrow \quad q \in \bigcup_{\mathring{c} \in \operatorname{dom} \mathring{b}} \left([\mathring{b}(\mathring{c})] \cap F^-(\mathring{c}\widehat{\in}\emptyset) \right) \\ &\Leftrightarrow \quad \exists \mathring{c} \in \operatorname{dom} \mathring{b} \left(\exists r \in \operatorname{ran} \mathring{b} \, [(\mathring{c}, r) \in \mathring{b} \wedge q \leq r \wedge q \in F^-(\mathring{c}\widehat{\in}\emptyset)] \right) \\ &\Leftrightarrow \quad q \leq e \wedge q \in F^-(\emptyset\widehat{\in}\emptyset) \\ &\Leftrightarrow \quad q \leq e \end{aligned}$$

because $F^-(\emptyset\widehat{\in}\emptyset) = \mathbb{P}$.

Suppose now that $r \in G \cap F^+(\emptyset\widehat{\subset}\emptyset)$. This means that $r \in F^-(\emptyset\widehat{\in}\emptyset)^\perp = e^\perp$. Since e is predense below p, this means that $r \perp p$, which is a contradiction. By truth-genericity, $G \cap F^-(\emptyset\widehat{\subset}\emptyset)$, which means that there is some $q \in G$ with $q \leq e$, so $G \cap e \neq \emptyset$. □

2.5 Splitting and Embedding

The basic scenario for this section is a pair of partial orders $\mathbb{P} \subset^* \mathbb{Q}$. Such partial orders should, it seems reasonable, give rise to the same extensions and the same notions of genericity, as is true in the set case. One cannot make this claim without additional restrictions on the orders, though. The problem is that (unlike with set forcing) $\mathbb{Q}$-names might not translate to $\mathbb{P}$-names; so to that end, we introduce the workhorse lemmas for such translations, followed by a notion that will let us take advantage of it.

Lemma 2.36 (Splitting Lemma)**.** *Suppose that* $\mathring{a} = \{(\mathring{a}'_\alpha, p_\alpha) : \alpha < \kappa\}$ *and that* $e_\alpha \subset [p_\alpha]$ *is predense below* p_α *for each* α*. Let* $\mathring{b} = \{(\mathring{a}'_\alpha, q_\alpha) : q_\alpha \in e_\alpha \wedge \alpha < \kappa\}$*. Then*

1. $F^-(\mathring{a}\widehat{\subset}\mathring{b}) = F^-(\mathring{b}\widehat{\subset}\mathring{a}) = \emptyset$

2. $F^+(\mathring{a}\widehat{=}\mathring{b}) = \mathbb{P}$

3. $\mathring{a}_G = \mathring{b}_G$ *for every generic* G

Proof. One can demonstrate the proposition for ordinary forcing as follows: let G be generic. For each α, $p_\alpha \in G$ iff $G \cap e_\alpha \neq \emptyset$, and hence $\mathring{a}_G = \mathring{b}_G$. Since generics are comeager, it follows that $\Vdash \mathring{a} = \mathring{b}$.

To get the more precise results, consider first $F^-(\mathring{a}\widehat{\subset}\mathring{b})$. This is given by

$$F^-(\mathring{a}\widehat{\subset}\mathring{b}) = \bigcup_{\mathring{a}' \in \mathrm{dom}(\mathring{a})} [\mathring{a}(\mathring{a}')] \cap F^-(\mathring{a}'\widehat{\in}\mathring{b})$$

Let $\mathring{a}' \in \mathrm{dom}(\mathring{a})$ be arbitrary and consider $F^-(\mathring{a}'\widehat{\in}\mathring{b})$. Since $\mathring{a}' \in \mathrm{dom}(\mathring{b})$ by construction, we have $F^-(\mathring{a}'\widehat{\in}\mathring{b}) \supset \mathring{b}(\mathring{a}')^\perp = \mathring{a}(\mathring{a}')^\perp$ by Lemma 2.29 and the fact that $\mathring{b}(\mathring{a}')$ is predense below $\mathring{a}(\mathring{a}')$ again by construction. It then follows that

$$[\mathring{a}(\mathring{a}')] \cap F^-(\mathring{a}'\widehat{\in}\mathring{b}) \subset [\mathring{a}(\mathring{a}')] \cap \mathring{a}(\mathring{a}')^\perp = \emptyset$$

proving the first equality. The second is proven in an identical fashion, making use of the fact that $[\mathring{b}(\mathring{a}')] \cap \mathring{a}(\mathring{a}')^{\perp}$ is again the empty set by predenseness. The other conclusions are trivial consequences of the first. □

The Splitting Lemma means that we are free to split the conditions in names as we see fit, provided we do so in a predense fashion. The power lies in the arbitrary nature of the splitting; in particular, it allows us to prove the following:

Lemma 2.37 (Regularization Lemma)**.** *Let $\mathring{a} \in V^{\mathbb{P}}$ and suppose that, for each $p \in \operatorname{supt}(\mathring{a})$, we specify a predense $e_p \subset [p]$. Then there is a name $\mathring{b}$ such that*

1. $F^{-}(\mathring{a}\widehat{\subset}\mathring{b}) = F^{-}(\mathring{b}\widehat{\subset}\mathring{a}) = \emptyset$
2. $\operatorname{supt}(\mathring{b}) = \bigcup e_p$
3. $\mathring{a}_G = \mathring{b}_G$ *for all generic filters* G

We call such a $\mathring{b}$ a regularization *of $\mathring{a}$ via the map $p \mapsto e_p$.*

Remark 2.38. *Although it would be nice if such a regularization were unique, this will depend on the combinatorics of $\mathbb{P}$ and the chosen predense sets. It is, however, "unique enough" for our purposes, and as such we will often refer to $\mathring{b}$ as* the *regularization of $\mathring{a}$, invariably constructed as in the proof below.*

Proof. Define $\mathring{b}$ by recursion, replacing each $p \in \operatorname{supt}(\mathring{a})$ by e_p. To be formal about it, define a map $\eta : \operatorname{domcl}(\mathring{a}) \to V^{\mathbb{P}}$ recursively by

$$\eta(\mathring{c}) = \{(\eta(\mathring{d}), r) : \mathring{d} \in \operatorname{dom}\mathring{c} \wedge \exists p[(\mathring{a}, p) \in \mathring{c} \wedge r \in e_p]\}$$

and letting $\mathring{b} = \eta(\mathring{a})$. This is exactly the same map as in the Splitting Lemma, simply applied hereditarily to $\mathring{a}$. We clearly have $Y = \bigcup e_p$; to show the relevant equalities, we will actually prove the stronger

$$F^-(\mathring{c}\widehat{\subset}\eta(\mathring{c})) = F^-(\eta(\mathring{c})\widehat{\subset}\mathring{c}) = \emptyset$$

for every $\mathring{c} \in \operatorname{domcl}(\mathring{a})$.

The proof is similar to that of the Splitting Lemma. Suppose the theorem is true for all $\mathring{d} \in \operatorname{dom}(\mathring{c})$. First, note that

$$F^+(\mathring{d}\widehat{\in}\eta(\mathring{c})) = \bigcup_{\eta(\mathring{d}')\in\operatorname{dom}(\eta(\mathring{c}))} [\{\eta(\mathring{c})\}(\eta(\mathring{d}'))] \cap F^+(\mathring{d}\widehat{=}\eta(\mathring{d}'))$$

Now $F^+(\mathring{d}\widehat{=}\eta(\mathring{d}')) = F^+(\mathring{d}\widehat{=}\mathring{d}') \cap F^+(\mathring{d}'\widehat{=}\eta(\mathring{d}')) = F^+(\mathring{d}\widehat{=}\mathring{d}')$ by induction. Furthermore, $\{\eta(\mathring{c})\}(\eta(\mathring{d}')) = \varepsilon(\mathring{c}(\mathring{d}'))$ where $\varepsilon(p) = e_p$ and is extended to sets of conditions in the obvious manner. This therefore reduces to:

$$F^-(\mathring{d}\widehat{\in}\eta(\mathring{c})) = \bigcup_{\mathring{d}'\in\operatorname{dom}(\mathring{c})} [\varepsilon(\mathring{c}(\mathring{d}'))] \cap F^+(\mathring{d}\widehat{=}\mathring{d}')$$

and so

$$\begin{aligned} F^-(\mathring{c}\widehat{\subset}\eta(\mathring{c})) &= \bigcup_{\mathring{d}\widehat{\in}\mathring{c}} [\mathring{c}(\mathring{d})] \cap F^-(\mathring{d}\widehat{\in}\eta(\mathring{c})) \\ &= \bigcup_{\mathring{d}\widehat{\in}\operatorname{dom}(\mathring{c})} [\mathring{c}(\mathring{d})] \cap \left[\bigcup_{\mathring{d}'\in\operatorname{dom}(\mathring{c})} [\varepsilon(\mathring{c}(\mathring{d}'))] \cap F^+(\mathring{d}\widehat{=}\mathring{d}') \right]^\perp \end{aligned}$$

Fix now $\mathring{d} \in \operatorname{dom}(\mathring{c})$ and consider what happens if $\mathring{d}' = \mathring{d}$ in the summand above. Then $F^+(\mathring{d}\widehat{=}\mathring{d}') = \mathbb{P}$, so this becomes $[\mathring{c}(\mathring{d})] \cap [\varepsilon(\mathring{c}(\mathring{d}))]^\perp = \emptyset$ because e_p is predense below p for every $p \in \operatorname{cond}(\mathring{c})$. Since each summand is empty, the union as a whole is empty, proving the claim. The other case is entirely similar and will be left to the reader. □

The simplest case in which one can apply these lemmas are when $\mathbb{Q}$ splits nicely over $\mathbb{P}$:

Definition 2.39. *If $\mathbb{P} \subset \mathbb{Q}$ then we say that $\mathbb{Q}$* is set-splitting *over $\mathbb{P}$ iff given any element $q \in \mathbb{Q}$, there is an $e \subset \mathbb{P}$ such that $e \subset [q]$ is predense below q. We sometimes say that e is a* (set-)splitting *of q in $\mathbb{P}$.*

As an aside, note that if one is worried about the use of Choice, one can pick a canonical splitting of q: let $e = V_\alpha \cap [q]$ where α is chosen minimal so that e is predense.

We now state and prove the Dense Embedding Theorems:

Theorem 2.40. *Suppose $(V; \mathbb{P}, \mathbb{Q}) \models \mathrm{ZFC}$, $\mathbb{P} \subset^* \mathbb{Q}$ and $\mathbb{Q}$ is set-splitting over $\mathbb{P}$. Then there is a natural correspondence between truth-generic filters $G \subset \mathbb{P}$ and $H \subset \mathbb{Q}$ given in the obvious manner*

$$\begin{aligned} G &\mapsto G{\uparrow}_{\mathbb{Q}} \\ H \restriction \mathbb{P} &\leftarrow\!\shortmid H \end{aligned}$$

and furthermore $(V; \mathbb{P}, \mathbb{Q})[G] = (V; \mathbb{P}, \mathbb{Q})[H]$.

Proof. Trivially, if $\mathring{a} \in V^{\mathbb{P}}$ then $\mathring{a}_G = \mathring{a}_H$ by a direct computation. For the other direction, let $\mathring{b} \in V^{\mathbb{Q}}$ be given, and let $\mathring{a}$ be the regularization of $\mathring{b}$ via the splittings of $\mathrm{supt}(\mathring{b})$ in $\mathbb{P}$. Then, by the Regularization Lemma, $\mathring{b}_G = \mathring{b}_H = \mathring{a}_H$ as required. □

We can say more:

Theorem 2.41. *Suppose $(V; \mathbb{P}, \mathbb{Q}) \models \mathrm{ZFC}$, $\mathbb{P} \subset^* \mathbb{Q}$ and $\mathbb{Q}$ is set-splitting over $\mathbb{P}$. Let φ be a sentence in the $\mathbb{P}$-forcing language, and also regard it as a sentence of the $\mathbb{Q}$-forcing language. Then*

$$F^+_{\mathbb{P}}(\varphi) = F^+_{\mathbb{Q}}(\varphi) \restriction \mathbb{P}$$

Hence if $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}} \varphi$ iff $p \Vdash_{\mathbb{Q}} \varphi$.

This will be proven once we have established the following two lemmas.

Lemma 2.42. *Given any quantifier-free sentence φ using $\mathring{a}, \mathring{b} \in V^{\mathbb{P}}$, i.e. $\mathbb{P}$-names only, $F^{\pm}_{\mathbb{Q}}(\varphi) \upharpoonright \mathbb{P} = F^{\pm}_{\mathbb{P}}(\varphi)$. In other words:*

1. *Any $p \in \mathbb{P}$ with $p \in F^{+}_{\mathbb{P}}(\mathring{a} \widehat{\in} \mathring{b})$ is also in $F^{+}_{\mathbb{Q}}(\mathring{a} \widehat{\in} \mathring{b})$ (and other quantifier-free sentences* mutatis mutandis*);*

2. *Any $q \in \mathbb{Q}$ in $F^{+}_{\mathbb{Q}}(\mathring{a} \widehat{\in} \mathring{b})$ has a p below it also in $F^{+}_{\mathbb{Q}}$.*

Proof. The second part of the lemma is trivial since F is always closed downwards and $\mathbb{P} \subset^{*} \mathbb{Q}$, so we need only show that $F^{+}_{\mathbb{P}}$ and $F^{+}_{\mathbb{Q}}$ are computed in the same way on these basic sets. This will naturally proceed by induction: suppose that the lemma is true for all $\mathring{a}, \mathring{b}$ sufficiently small; then, for the first of the inductive arguments:

$$\begin{aligned} p \in F^{+}_{\mathbb{Q}}(\mathring{a} \widehat{\in} \mathring{b}) &\Leftrightarrow p \in \bigcup_{\mathring{c} \in \mathrm{dom}(\mathring{b})} \Big([\mathring{b}(\mathring{c})] \cap F^{+}_{\mathbb{Q}}(\mathring{a} \widehat{\subset} \mathring{c}) \cap F^{+}_{\mathbb{Q}}(\mathring{c} \widehat{\subset} \mathring{a}) \Big) \\ &\Leftrightarrow p \in \bigcup_{\mathring{c} \in \mathrm{dom}(\mathring{b})} \Big([\mathring{b}(\mathring{c})] \cap F^{+}_{\mathbb{P}}(\mathring{a} \widehat{\subset} \mathring{c}) \cap F^{+}_{\mathbb{P}}(\mathring{c} \widehat{\subset} \mathring{a}) \Big) \\ &\Leftrightarrow p \in F^{+}_{\mathbb{P}}(\mathring{a} \widehat{\in} \mathring{b}) \end{aligned}$$

where the middle implication follows from the induction hypothesis and the fact that $\mathring{b} \in V^{\mathbb{P}}$, not just $V^{\mathbb{Q}}$. A similar argument works for $F^{-}(\mathring{a} \widehat{\subset} \mathring{b})$, and the negation and conjunction follows from the fact that interchapter and $\perp$ play nice with dense subsets. □

To tackle the quantification step, we use the Regularization Lemma:

Lemma 2.43. *Suppose* $\mathring{b} \in V^{\mathbb{Q}}$ *and let* $\mathring{c} \in V^{\mathbb{P}}$ *be its regularization. Then, for any* $\mathring{a} \in V^{\mathbb{P}}$

1. $F^+_{\mathbb{P}}(\mathring{c} \in \mathring{a}) \subset^* F^+_{\mathbb{Q}}(\mathring{b} \in \mathring{a})$
2. $F^-_{\mathbb{P}}(\mathring{c} \in \mathring{a}) = F^-_{\mathbb{Q}}(\mathring{b} \in \mathring{a}) \upharpoonright \mathbb{P}$
3. $F^-_{\mathbb{P}}(\mathring{c} \subset \mathring{a}) \subset^* F^-_{\mathbb{Q}}(\mathring{b} \subset \mathring{a})$
4. $F^+_{\mathbb{P}}(\mathring{c} \subset \mathring{a}) = F^+_{\mathbb{Q}}(\mathring{b} \subset \mathring{a}) \upharpoonright \mathbb{P}$

Proof. We will prove the first and third by double induction, the other two being immediate consequences. Consider the membership clause first:

$$\begin{aligned} F^+_{\mathbb{P}}(\mathring{c} \in \mathring{a}) &= \bigcup_{\mathring{a}' \in \mathrm{dom}(\mathring{a})} [\mathring{a}(\mathring{a}')]_{\mathbb{P}} \cap F^+_{\mathbb{P}}(\mathring{c} = \mathring{a}) \\ &= \bigcup_{\mathring{a}' \in \mathrm{dom}(\mathring{a})} [\mathring{a}(\mathring{a}')]_{\mathbb{P}} \cap F^+_{\mathbb{Q}}(\mathring{c} = \mathring{a}) \upharpoonright \mathbb{P} \\ &\subset^* \bigcup_{\mathring{a}' \in \mathrm{dom}(\mathring{a})} [\mathring{a}(\mathring{a}')]_{\mathbb{Q}} \cap F^+_{\mathbb{Q}}(\mathring{c} = \mathring{a}) \\ &\subset^* F^+_{\mathbb{P}}(\mathring{c} \in \mathring{a}) \end{aligned}$$

The other clause is entirely similar.

□

Corollary 2.44. *Suppose* $\mathring{a} \in V^{\mathbb{P}}$*. Then*

$$\bigcup_{\mathring{c} \in V^{\mathbb{P}}} F^+_{\mathbb{P}}(\mathring{c} \widehat{\in} \mathring{a}) = \bigcup_{\mathring{b} \in V^{\mathbb{Q}}} F^+_{\mathbb{Q}}(\mathring{b} \widehat{\in} \mathring{a}) \upharpoonright \mathbb{P}$$

Hence forcing coheres for dense suborders over which the original order setsplits.

2.6 Dense embedding for definably-generic forcings

The logical question to ask at this point is whether set-splitting is required for this relationship to hold. In order to talk about this more easily, we make the following definition:

Definition 2.45. *A combinatorial property (C)* respects density *if it can replace set-splitting in the hypotheses of the Dense Embedding Theorems (Theorems 2.40 and 2.41). That is, if (C) holds relative to $\mathbb{P} \subset^* \mathbb{Q}$, then generics over $\mathbb{P}$ and $\mathbb{Q}$ correspond in the usual way, and forcing over $\mathbb{P}$ and $\mathbb{Q}$ are the same.*

There is a very nice characterization of definably-generic filters which respect density, although it is unproven in any of the standard literature.[2] The central notion is that of "pretameness" (c.f. [2]) or "predensity reduction" (c.f. [5]). As we will be trying to extend these concepts to truth-generic filters, we will give the definition in two stages.

Definition 2.46. *Let $\mathbb{P}$ be a poclass and $\langle S_i : i \in I \rangle$ be a uniformly enumerated sequence of classes with $I \in V$. We say that $p \in \mathbb{P}$* reduces *or* localizes *the sequence S_i precisely if there is a sequence $\langle s_i : i \in I \rangle \in V$ such that*

1. *$s_i \subset S_i$ for each $i \in I$*
2. *s_i is predense below p.*

The idea behind the name is that the classes S_i, which are a global phenomenon, can be reduced to predense sets locally, i.e. underneath p. Pretameness then becomes:

[2] We make no claims to originality, of course; it's simply that the proof itself is not explicit in any of the standard works.

Definition 2.47. $\mathbb{P}$ *is pretame iff for every uniform sequence of dense classes* $\langle D_i : i \in I\rangle$, *the class* $\{p \in \mathbb{P} : p$ *reduces all the* $D_i\}$ *is dense in* $\mathbb{P}$.

We now prove the following theorem, which is implicit (though unproven) in Friedman.

Theorem 2.48. *Suppose that* $(V; \mathbb{P}, \mathbb{Q}) \models \mathrm{ZFC}$ *and* $\mathbb{P} \sqsubset (V; \mathbb{Q})$. *Suppose further that* $\mathbb{P} \subset^* \mathbb{Q}$ *and that* $\mathbb{Q}$ *is pretame. Then there is a natural correspondence between definably generic filters* G *on* $\mathbb{P}$ *and* H *on* $\mathbb{Q}$ *such that* $V[G] = V[H]$ *for all such* G/H, *and* $\Vdash_{\mathbb{P}} = \Vdash_{\mathbb{Q}} \restriction \mathbb{P}$. *In other words, pretameness respects density for definably-generic forcing.*

Proof. Trivially $V^{\mathbb{P}} \subset V^{\mathbb{Q}}$, so we get $V[G] \subset V[H]$ for free. In the set forcing case this suffices, as we can define a $\mathbb{P}$-name for H and use minimality to show the reverse inclusion, but since G and H are classes, they cannot be named by any set name in the ground model. We must therefore show directly that for any $\mathbb{Q}$-name $\mathring{b} \in V^{\mathbb{Q}}$ and any H $\mathbb{Q}$-generic we have a $\mathbb{P}$-name $\mathring{a} \in V^{\mathbb{P}}$ such that $\mathring{b}_H = \mathring{a}_G$.

To that end, we will employ two density arguments: the first, via pretameness, to demonstrate that certain classes of conditions are dense in $\mathbb{P}$ and the second, via genericity, to demonstrate that (at least one) of a particular class of names evaluates correctly. Given any element $q \in \mathbb{Q}$, define the *dense set generated by* q, $\mathbb{Q}/q$, to be $\{q' \in \mathbb{Q} : q' \leq q \vee q' \perp q\}$, i.e. those conditions either below or incompatible with q. Let $X = \{q_i : i \in I\} \subset \mathbb{Q}$, and consider the sequence of dense classes $D_i = \mathbb{Q}/q_i \cap \mathbb{P}$. By pretameness, given any $q \in \mathbb{Q}$ we can find a $q' \leq q$ such that each of the D_i reduces to a d_i pretame below q'.

Consider now the collection

$$\begin{aligned} E_X &= \{q \in \mathbb{Q} : \exists d[\forall i{\in}I\ (d_i \subset D_i \wedge\ d_i \text{ predense} \leq q\ )]\} \\ &= \{q \in \mathbb{Q} : \exists d[\forall i{\in}I\ (d_i \subset \mathbb{Q}/q_i \wedge d_i \subset \mathbb{P} \wedge\ d_i \text{ predense} \leq q\ ]\} \end{aligned}$$

This is a definable subclass of $\mathbb{Q}$, it is of constant complexity (because the map $q_i \mapsto D_i$ is uniform) and is dense by pretameness as above. This means that there is a *uniform* map $\mathcal{E} : X \mapsto E_X$ which we can use to regularize $\mathbb{Q}$-names into $\mathbb{P}$ names; but it does not quite meet the specifications of the Regularization Lemma. Instead, it follows from the Local Regularization Lemma, proven below. specifically, apply the Local Regularization Lemma to $\mathring{b}$ relative to q via the map $q_i \mapsto d_i$ to produce $\mathring{a}$. □

Lemma 2.49 (Local Regularization Lemma)**.** *Fix $p_0 \in \mathbb{P}$ and let $\mathring{a} \in V^{\mathbb{P}}$. Suppose that, for each $p \in \text{supt}(\mathring{a})$, we specify a predense $e_p \subset [p_0] \cap \mathbb{P}/p$. Then there is a name $\mathring{b}$ such that*

1. $F^+(\mathring{a} = \mathring{b}) = [p_0]$
2. $\text{supt}(\mathring{b}) = \bigcup e_p$
3. $\mathring{a}_G = \mathring{b}_G$ *for all generic filters $G \ni p_0$*

We say that this regularization is relative to q*.*

Remark 2.50. *Note that the Regularization Lemma is a special case of the Local Regularization Lemma, taking $p_0 = \mathbf{1}$ and $e_p \subset \mathbb{P}$.*

Proof. The first step is to split the e_p into e_p^+ and e_p^-, the conditions which are either below or incompatible to p respectively. We now regularize $\mathring{a}$ relative to the map $p \mapsto e_p^+$;

in other words, $\mathring{b}$ is given by replacing all occurrences of p only by those conditions below p. We now confirm that this $\mathring{b}$ has the requisite properties. Rather than go through the combinatorics, which are quite ugly, we will instead use the semantics. Let G be a generic filter, and suppose that $p_0 \in G$. Then for every $d \in \mathring{c}_G$, $(\mathring{d}, p) \in \mathring{c}$ for some $\mathring{d}_G = d$ and some $p \in G$. Then $\mathring{d}_G = \eta(\mathring{d})_G$ by induction, while $(\eta(\mathring{d}), q) \in \eta(\mathring{c})$ for every $q \in e_p^+$ by construction. Now e_p is predense below p_0, so $G \cap e_p \neq \emptyset$, but $G \cap e_p^- = \emptyset$ because $p \in G$, so $G \cap e_p^+ \neq \emptyset$ and hence $\mathring{d}_G = \eta(\mathring{d})_G \in \eta(\mathring{c})_G$. This proves that $\mathring{c}_G \subset \eta(\mathring{c}_G)$, and the other direction is similar.

□

2.7 Generalizing to truth-genericity

The distinction between definable-genericity and truth-genericity is that one only needs – and indeed, is only guaranteed – to intersect all dense sets of the form $F(\varphi)$. To generalize this argument, then, requires finding such a sentence that can encapsulate the idea of localizing elements.

To that end, let us consider the collection $D_i = \mathbb{Q}/q_i$ defined above. A simple application of Lemma 2.27 yields the following incredibly fruitful observation:

$$\mathbb{Q}/q_i = F(q_i^{\Gamma} \widehat{\subset} \emptyset)$$

Recall that $F(\varphi) = F^+(\varphi) \cup F^-(\varphi)$; this suggests that any sentential rendition of localization should be split into two parts, one expressing predensity in F^+ and the other in F^-. Accordingly, consider the following sentences:

1. $\varphi_1 \equiv \neg(\mathring{e}_1 \cup \mathring{e}_2 \widehat{\subset} \emptyset)$

2. $\varphi_2 \equiv \mathring{e}_1 \widehat{\subset} \emptyset \leftrightarrow q^\Gamma \widehat{\subset} \emptyset$

3. $\varphi_3 \equiv \mathring{e}_2 \widehat{\subset} \emptyset \leftrightarrow \neg(q^\Gamma \widehat{\subset} \emptyset)$

Proposition 2.51. *Given φ_i as above, define $e^+ = \mathrm{cond}(\mathring{e}_1)$ and $e^- = \mathrm{cond}(\mathring{e}_2)$. Then the following hold:*

1. $p \in F^+(\varphi_1)$ implies $e^+ \cup e^-$ is predense below p.

2. $p \in F^+(\varphi_2)$ implies $[p] \cap [e^+] = [p] \cap [q]$ are dense in each other.

3. $p \in F^+(\varphi_3)$ implies $[p] \cap [e^-] = [p] \cap [q]^\perp$ are dense in each other.

Proof. These all follow directly from Lemma 2.27 and the modified definition of F on disjunctions. By way of illustration, consider $p \in F^+(\mathring{e}_1 \widehat{\subset} \emptyset \rightarrow q^\Gamma \widehat{\subset} \emptyset) = F^-(\mathring{e}_1 \widehat{\subset} \emptyset) \cup F^+(q^\Gamma \widehat{\subset} \emptyset) = [e^+] \cup [q]^\perp$. Hence $[p] \cap [q] \subset [p] \cap [e^+]$ and the reverse inclusion is similar. □

Corollary 2.52. *If $p \in F^+(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)$ then p reduces $\mathbb{Q}/q$.*

Now define

$$\varphi(\mathring{e}_1, \mathring{e}_2, \kappa) \equiv \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge (\mathring{e}_1 \cup \mathring{e}_2) \widehat{\subset} (V^{\mathbb{P}} \cap V^\kappa)\check{}$$

The parameters $\mathring{e}_1, \mathring{e}_2, \kappa$ are not really parameters of a single φ; rather, they parametrize a family of φ, one for each triple. Note also that $V^{\mathbb{P}} \cap V^\kappa$ is a set, which we will call $V^{\mathbb{P}}_\kappa$, and hence (after the usual embedding) a constant in the language $\mathcal{L}^{\mathbb{P}}$.

Theorem 2.53. *Let $\mathring{a} \in V^{\mathbb{Q}}$ and let φ_i be a sentence of the type given above for every $q_i \in \mathrm{supt}(a)$. Suppose $q \in F^+(\varphi_i)$ for every such φ_i. Then there is a name $\mathring{b} \in V^{\mathbb{P}}$ such that $q \in F^+(\mathring{a} \widehat{=} \mathring{b})$.*

Remark 2.54. *We have changed the* p *from the earlier theorem to* q *here as that is how it will be used in the sequel.*

Proof. As before, let $e_i^+ = \mathrm{cond}(\mathring{e}_i^1)$ and similarly for e_i^-. By the corollary, q reduces $\mathbb{Q}/q_i \cap \mathbb{P}$ for every i; hence the proof of Lemma 2.37 goes through as before. □

Theorem 2.55. *Under the hypotheses of the preceding theorem, let*

$$\varphi^\kappa_{\text{dense}} \equiv \exists e : I \times 2 \to \mathcal{P}(V^{\mathbb{P}}_\kappa)[\forall i \in I(\varphi(e(i,0), e(i,1)), \kappa)]$$

where φ *is as above and* κ *is fixed. If* $q \in F^+(\varphi^\kappa_{\text{dense}})$ *and* $G \ni q$ *is generic, then there is a name* $\mathring{b} \in V^{\mathbb{P}}$ *such that* $q \Vdash \mathring{a} = \mathring{b}$.

Proof. The set of individual witnesses to the existential are dense below q, and by the preceding theorem they all force $\mathring{a} = \mathring{b}$. □

We are now in a position to provide a combinatorial condition that respects density for truth-generic forcing:

Theorem 2.56. *The condition "*$\mathbf{1} \Vdash \varphi^\kappa_{\text{dense}}$*" respects density.*

Assuming we have truth-genericity for the language augmented by a predicate for the partial order, we can create a similar sentence φ_{dense} which will work without a uniform restriction on the size of the witnesses for the splitting.

2.8 Uses of pretameness in definably-generic forcings

The following theorems are sketched in outline in Friedman; we amplify them here to motivate (we hope) the definitions for truth-generic forcing. None of the arguments here are original in the slightest.

Proposition 2.57. *Suppose that $\mathbb{P}$ is V-amenable and for each $p \in \mathbb{P}$ there is a $G \ni p$ such that G is definably generic over $(V;\mathbb{P})$ and $(V[G];V,\mathbb{P},G) \models \mathrm{ZF}-\mathrm{P}$. Then $\mathbb{P}$ is pretame.*

Proof. We prove the contrapositive: suppose that $p \in \mathbb{P}$, $\langle D_i : i \in I\rangle \sqsubset V$ witness the failure of pretameness; we show that Replacement fails in $(V[G];V,\mathbb{P},G)$ where $G \ni p$ is definably generic over $(V;\mathbb{P})$. Specifically, let F be the predicate

$$F(i) = \mu\alpha[G \cap D_i \cap V_\alpha \neq \emptyset]$$

in other words, $F(i)$ is the least α such that $G \cap D_i$ is non-empty. $F(i)$ exists because G, being definably generic, intersects all the D_i, and F itself is $(V[G];V,\mathbb{P},G)$ definable. Further, all pieces are necessary: G is obviously needed in the definition of F and we need V and $\mathbb{P}$ to define the D_i. This F is thus a class map $F : I \to \infty$ which, by Replacement, should be bounded in $(V[G];V,\mathbb{P},G)$ since $I \in V$.

Let $q \leq p$. We can regard WLOG any predense $d_i \subset D_i$ as being $D_i \cap V_\alpha$ for some α. Hence, since p is not pretame, q cannot simultaneously reduce all the D_i, which means that for any α there is some i such that $D_i \cap V_\alpha$ is not predense below q. This in turn means that, for any α there is some i and some $r \leq q$ such that $r \perp D_i \cap V_\alpha$. This means that, for each α, the following is dense below p:

$$E_\alpha = \{r : \exists i{\in}I\ (r \perp D_i \cap V_\alpha)\}$$

Since $p \in G$ and G is definably generic, for each α there is an $r_\alpha \in E_\alpha \cap G$, which means that $r_\alpha \perp D_i \cap V_\alpha$. Suppose now that $f(i) < \beta$ for every $i \in I$. This means that, for each i, there is $q \in G \cap D_i \cap V_\beta$. Consider r_β: by definition, there is some i such that $r \perp D_i \cap V_\beta$, whence $r \perp q$ which is a contradiction. $\square$

Careful observation will yield the following important facts:

- We only need a $G \ni p$ definably generic for every p which fails to reduce some sequence of D_i.
- We can break Replacement with a single predicate...
- ...but one which depends, in an essential manner, on being able to exploit V, $\mathbb{P}$ and G.

Unfortunately, at this point, nothing further can be stated.

Chapter 3

Boolean Algebras and the Definability of Forcing

A standard way of attacking the problem of defining forcing is to go about it indirectly. That is, rather than defining forcing for the order directly by using a complicated double-recursion argument (see Chapter 4), we embed the order into its boolean completion, where forcing has a much simpler (and arguably more natural) definition. The key facts to be established in this process are:

1. Forcing can be easily defined over boolean algebras
2. Every partial order has a boolean completion into which it can be embedded
3. Forcing over the completion is the same as forcing over the original order

Moving to the class level, it is clear that much work must be done before these can be demonstrated. Crucial in this process is the question of the relationship between generic filters over the larger order and generic filters over the smaller order; though the relationship is obvious when one deals with sets or definable genericity, it is a much subtler question when dealing with truth-genericity.

3.1 Forcing over boolean algebras

While the notion of a "complete boolean algebra" is not particularly clear when talking of class orders – complete with respect to what? – it turns out that there is one condition that clearly suffices.

Definition 3.1. *A boolean algebra* $\mathbb{B}$ *is said to be a* set-complete boolean algebra *if it is closed under sups and infs of arbitrary sets of elements.*

Theorem 3.2. *If* $\mathbb{B}$ *is a set-complete boolean algebra, then* $\Vdash_{\mathbb{B}}$ *is definable.*

Proof. The key is, very simply, that the structure of the boolean algebra allows the reduction of the classes in Definition 2.2 to actual elements of $\mathbb{B}$ on which ordinary recursion can operate. Specifically, we will show that given any quantifier-free sentence φ of the forcing language, there is an element $b_\varphi \in \mathbb{B}$ such that $\mathrm{reg}(F^{\pm}(\varphi)) = [b_\varphi]$, and that such b_φ may be (first-order) recursively defined from φ.

Assume that this has been done already by induction. Negation and conjunction are trivial, being represented by the boolean complement and meet of $\mathbb{B}$, so all that remains is to show that membership and subset are represented in this fashion. Note first that if $p_i \subset \mathbb{P}$ for $i \in I \in V$, then

$$\mathrm{reg}(\bigcup [p_i]) = [\bigvee p_i]$$

In a slightly more complex fashion, assuming that $p_i, q_j \subset \mathbb{P}$, note that

$$\mathrm{reg}(\bigcup [p_i] \cap [q_j]) = \mathrm{reg}(\bigcup [p_i^\alpha \wedge q_j^\beta]) = [\bigvee (p_i^\alpha \wedge q_j^\beta)]$$

with obvious generalizations to set-many interchapters. We can therefore identify the correct element $b_\varphi \in \mathbb{B}$ such that $[b_\varphi] = \mathrm{reg}(F^{\pm}(\varphi))$ in a first-order fashion via the usual recursion theorem, and quantifier-free forcing can then be defined as: $p \Vdash \varphi$ iff $p \leq b_\varphi$. We can then extend to the quantified forcing in the usual way. $\square$

The obvious advantage of the boolean algebraic setting is that, as just shown, for any sentence φ there is a single element b_φ which completely encapsulates its truth; indeed, it is sometimes called the "truth value" $[\varphi]$. Here, we only have exact truth values for the quantifier-free sentences because of the class-large union required for quantifiers, but that suffices. The crux, of course, is that we let the structure of the BA take over the complexity of the definition for us. Order-theoretic $\perp$ becomes the simple BA complement, and the regular closure becomes the simple BA join. For classes, however, it is not clear whether there is a boolean completion or whether such completions give rise to the same notions of forcing, and so it is to those matters we turn.

3.2 Extending orders to BAs

The basic theorem of this chapter is:

Theorem 3.3. *If $\mathbb{P}$ is V-amenable, there is a $\overline{\mathbb{P}} \sqsubset (V; \mathbb{P})$ such that*

1. *$\overline{PP}$ is a Boolean algebra*
2. *$\mathbb{P}$ densely embeds into $\overline{PP}$*
3. *$\overline{PP}$ is set-complete over V; that is, every $X \subset \overline{PP}$ with $X \in V$ has a sup and an inf in $\overline{PP}$.*
4. *$\mathbb{P}$ and $\overline{PP}$ are interdefinable over V.*

Furthermore, this completion is unique, in the sense that any complete embedding of $\mathbb{P}$ *into a set-complete boolean order extends to a unique complete embedding of* $\overline{PP}$.

Proof. We do this by constructing the "set regular open algebra" of $\mathbb{P}$ and showing that it has the requisite properties, which we will do in stages. Each stage $\mathbb{P}_\alpha$ will consist of "set-completing" the orders that came before it, with $\overline{PP} = \bigcup \mathbb{P}_\alpha$ being a set-complete boolean algebra by virtue of the axiom of replacement. Uniqueness will follow from the fact that every step of the construction can be replicated in the embedding.

Specifically, given $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ with $\mathbb{P}_\gamma \subset \mathbb{P}_\beta$ for $\gamma < \beta$ and any $A \subset \bigcup_{\beta<\alpha} \mathbb{P}_\beta$, define

$$
\begin{aligned}
[\wedge A] &= \{p : p \leq A\} = \{p : \forall q{\in}A \, (p \leq q)\} \\
[\vee A] &= \bigcup A \cup \{p : \forall r{\in}\mathbb{P} \, (r \geq A \rightarrow r \geq p) \wedge \forall r{\in}\mathbb{P} \, (r \perp A \rightarrow r \perp p)\} \downarrow \\
[\neg A] &= \{p : p \perp A\} = \{p : \neg \exists q{\in}A \, \exists r{\in}\mathbb{P} \, (r \leq p \wedge r \leq q)\}
\end{aligned}
$$

then let $\mathbb{P}_\alpha$ be the union of all such terms, ordered these by inclusion. To ensure interdefinability, we will tag each term with the level α at which they arose, though we will subsequently ignore this fact throughout. Finally, we will embed $\mathbb{P}_\beta$ into $\mathbb{P}_\alpha$ as downward cones, namely $p \mapsto [\wedge\{p\}]$; this will turn out to be equivalent to embedding it as $p \mapsto [\vee\{p\}]$, though this needs to be checked. As per usual, we write $[\wedge p]$ and $[\vee p]$ when the set X is a singleton.

Remark 3.4. *All of this is completely standard; it is essentially the regular closure restricted to "small" subsets, with a constructive definition of the meet chosen to replicate the standard, impredicative "least regular open set containing" construction of the regular open algebra. What is non-standard are the technicalities of the construction.*

To be precise, we will define by recursion a class $\mathbb{P}_\alpha = \mathbb{P}(\alpha, \cdot)$, defined as the equivalence classes (via Scott's trick) of a base preorder $\mathbb{P}'_\alpha$ with order $\leq_\alpha$, where $a \sim_\alpha b$ iff $a \leq_\alpha b \wedge b \leq_\alpha a$. An element of $\mathbb{P}'_\alpha$ will be triple (X, ϵ, α), where $\epsilon \in 3 \equiv \{\neg, \wedge, \vee\}$ and $X \subset \bigcup\{\mathbb{P}_\beta : \beta < \alpha\}$, which really means: $\forall x{\in}X\ \exists\gamma < \alpha[\mathbb{P}(\gamma, x)]$.

We now order these elements in a formalization of the informal descriptions above. For example, thinking of 1 as $\neg$ and 2 as $\wedge$, we have $(X, 1, \alpha) \leq_\alpha (Y, 2, \alpha)$ iff

$$\begin{aligned}
(X, 1, \alpha) \leq_\alpha (Y, 2, \alpha) \quad &\Leftrightarrow \quad \forall p{\in}\bigcup\{\mathbb{P}_\beta : \beta < \alpha\}\ (p \perp X \to p \leq Y) \\
&\Leftrightarrow \quad \forall \beta < \alpha \forall p{\in}\mathbb{P}_\beta\ (p \perp X \to p \leq Y) \\
&\Leftrightarrow \quad \forall \beta < \alpha[(\mathbb{P}(\beta, p) \wedge p \perp X) \to p \leq Y] \\
&\Leftrightarrow \quad \forall \beta < \alpha \forall x{\in}X\ \forall y{\in}Y\ [(\mathbb{P}(\beta, p) \wedge \mathbb{P}(\beta, x) \wedge \mathbb{P}(\beta, y) \wedge p \perp_\beta x) \\
&\qquad\quad \to (p \leq_\beta y)]
\end{aligned}$$

since, by construction, $x \leq_\beta y$ iff $x \leq_\gamma y$ whenever $\beta < \gamma$ and $x, y \in \mathbb{P}_\beta$. It is now a straightforward, though tedious, verification that $\leq_\alpha$ is a preorder. Once this has been established, we will return to the informal definition, remembering (in the words of Kanamori) that this is merely *un façon de parler*.

Remark 3.5. *Rather than writing out the tedious technical definition of the sup, we will sometimes paraphrase as follows:* $p \in [\vee X]$ *precisely if* p *is below everything above* X*, and so forth.*

To confirm that this all works, we must check that a number of conditions are witnessed:

1. $\mathbb{P}_\beta \to \mathbb{P}_\alpha$ is order-preserving, hence can be considered as an actual subclass.

2. $\mathbb{P}_\beta \subset^* \mathbb{P}_\alpha$.

3. $\mathbb{P}$ and $\bigcup\{\mathbb{P}_\alpha : \alpha < \infty\}$ are interdefinable.

4. Sups, infs and complements are added correctly at stage α.

5. Sups, infs and complements added at a previous stage $\beta < \alpha$ remain sups, infs and complements at stage α.

Only the third and fourth are non-trivial, so we will tackle them in stages. Let us first check that the cone $[\wedge p]$ and $[\vee p]$ are the same. $[\wedge p] = p\downarrow = \{q \in \mathbb{P} : q \leq p\}$. $[\vee p]$, on the other hand, is defined to be $p\!\downarrow \cup \{p : \forall r{\in}\mathbb{P}\,(r \geq A \rightarrow r \geq p) \wedge \forall r{\in}\mathbb{P}\,(r \perp A \rightarrow r \perp p)\}$. If $q \leq p$ then clearly $q \in [\vee p]$ so suppose that $q \not\leq p$. Since $\mathbb{P}$ is separative, there is some $r \leq q$ with $r \perp p$. This means that $q \notin [\vee p]$ as required.

To check the correctness of the sups and infs, we simply grind through the definitions. $[\wedge X]$ should be the least upper bound of all things below X, which is trivially true. $[\neg X]$ should be the maximal element incompatible with every element of X, which again is trivially true. Furthermore, it is clear that both infs and complements are preserved when going from $\mathbb{P}_\beta$ to $\mathbb{P}_\alpha$. For example, if $q = \bigwedge^{\mathbb{P}_\beta} X$, then clearly $q \in [\wedge X]$ (and so $[q] \subset [\wedge X]$), while if $r \in \bigwedge^{\mathbb{P}_\beta}\{[p] : p \in X\}$ then $r \leq_\beta X$ whence $q \geq_\beta r$ and so $r \in [q]$ as required.

The only tricky part – as, indeed, is the only tricky part of the usual construction – is to confirm that the sup was added correctly and didn't spoil existing sups. To do this, we must be quite careful about the nature of the elements that we added at stage α. Suppose then that $X, X' \subset \bigcup \mathbb{P}_\beta$; we must show that, for each of $[\vee X']$, $[\wedge X']$ and $[\neg X']$, if this new element dominates $[p]$ for each $p \in X$ and $q \in [\vee X]$, then q is in this new element. Throughout the following, p will be an element of X and q an element of $[\vee X]$, and $\leq$ (or $\perp$, *mutatis mutandis*) will refer to any of the $\leq_\beta$ ($\perp_\beta$), which all cohere by the induction hypothesis.

Suppose first that $[p] \subset [\neg X']$ for each $p \in X$. This means that $p \perp X'$ for each $p \in X$, or in other words $X \perp X'$. Since q is incompatible with everything incompatible with X, this in turn means that $q \perp X'$ whence $q \in [\neg X']$.

Suppose next that $[p] \subset [\wedge X']$ for each $p \in X$. This means that $p \leq X'$ for each $p \in X$, or in other words that $X \leq X'$. Since q is below everything above X, this in turn means that $q \leq X'$ whence $q \in [\wedge X']$.

Finally, suppose that $[p] \subset [\vee X']$ for each $p \in X$. This means that p is below everything that dominates X', and incompatible with everything incompatible with X'. The latter condition means that if $r \perp X'$ then $r \perp X$, which means that $q \perp r$ or q is incompatible with everything incompatible with X'. The former means that if $r \geq X'$ then $r \geq X$, which means that $q \leq r$ as q is below everything that dominates X, so that q is below anything that dominates X'. Put together, this implies that $q \in [\vee X']$ as required. $\square$

As a corollary, note the following:

Theorem 3.6. *If $\overline{\mathbb{P}}$ is set-splitting over $\mathbb{P}$, then $\Vdash_{\mathbb{P}}$ is definable.*

In order to apply Theorem 2.40, we need some kind of condition that determines when a partial order is set-splitting in its own completion. There does not seem to be a good, full characterization, but there is a sufficient condition of interest.

Theorem 3.7. *Let $\mathbb{P}$ be separative. Suppose that, given any $X, Y \subset \mathbb{P}$ with $X, Y \in \mathbb{V}$, there is a predense $Z \subset [X] \cap [Y]^{\perp}$ and a $W \subset \mathrm{reg}([X]^{\mathrm{op}} \cap ([Y]^{\perp})^{\mathrm{op}})$ predense in the reverse ordering on $\mathbb{P}$, with $Z, W \in \mathbb{V}$. Then $\overline{\mathbb{P}}$ set-splits over $\mathbb{P}$ and hence is $\Vdash_{\mathbb{P}}$ is definable.*

Proof. Clearly $\mathbb{P}$ is set-splitting over itself; we need only show that every $\mathbb{P}_\alpha$ is set-splitting over $\mathbb{P}$. The proof of this is utterly routine, except for the reverse ordering condition: as complementing reverses the order on $\mathbb{P}$ (i.e. $p \leq q$ iff $\neg p \geq \neg q$) we need to control not just predensity going up but predensity going down. $\square$

We note that any condition on $\overline{\mathbb{P}}$ and $\mathbb{P}$ that respects density will necessarily be symmetric in this way, because $\overline{\mathbb{P}}$ is symmetric up and down; more technically, complementation is an order-preserving involution.

Proof. Let us generalize the hypotheses of the theorem to the condition

$$\mathcal{R}_\alpha : \forall X, Y \subset \mathbb{P}_\alpha \exists Z \subset \mathbb{P}(Z \text{ is predense in } \mathrm{reg}([X] \cap [Y]^\perp))$$

and similarly for $\mathcal{R}_\alpha^{op}$. The theorem will then follow from $\mathcal{R}_\infty$ (considering $\overline{\mathbb{P}} = \mathbb{P}_\infty$) since that will trivially imply that $\overline{\mathbb{P}}$ is set-splitting over $\mathbb{P}$.

Since the situation is symmetric both upwards and downwards, we will restrict our attention to $\mathcal{R}_\alpha$, and to establish that $\mathcal{R}_\alpha$ holds for all $\alpha < \infty$, assume that $\mathcal{R}_\beta$ holds for all $\beta < \alpha$. To simplify the exposition, call the elements of X the "positive" conditions and those of Y the "negative" conditions; the idea is to composit the positive and negative conditions of the elements of $\mathbb{P}_\alpha$ into positive ($\mathcal{P}$) and negative ($\mathcal{N}$) conditions of $\mathbb{P}$, then applying the hypotheses of the theorem.

Specifically, we proceed as follows. Let $X, Y \subset \mathbb{P}_\alpha$ be given. If $p \in X$ is of the form $[\bigwedge A]$, add A to $\mathcal{P}_0$; if $p \in X$ is of the form $[\neg A]$, add the set W derived from A from the hypotheses to $\mathcal{N}_0$; and do the opposite for $q \in Y$. These form the basic positive and negative conditions. To handle positive joins and negative meets, note the following simple lemma:

Lemma 3.8. *Let $[A] \subset \mathbb{P}$ be given, and suppose that $B \subset [A]$ is predense. Suppose further that, for every $C \subset B$, there is a $D_C \subset [C]$ such that D_C is predense in $[C]$. Then $\bigcup D_C$ is predense in A.*

To that end, let M be the set of all $p \in X$ such that p is of the form $[\bigvee A_p]$. Now, for every choice f of a subset of A_p – in other words, for every $f : M \to \mathbb{V}$ such that $f(p) \subset A_p$ – define the set $\mathcal{P}_f = \mathcal{P} \cup \bigcup \text{ran}(f)$, which represent the elements compatible with exactly A_p. Do similarly for q of the form $[\bigwedge B_q]$ in the negative conditions, giving rise to $\mathcal{N}_g$ where g is a similar choice function $g : N \to \mathbb{V}$. We now apply the hypothesis to the sets $\mathcal{P}_f$ and $\mathcal{N}_g$ to produce a $Z_{fg} \subset \mathbb{P}$ that is predense below $[\mathcal{P}_f] \cup [\mathcal{N}_g]$. It is now routine, though tedious, to confirm that

$$Z = \bigcup \{Z_{fg} : f \in M \wedge g \in N\}$$

is the required predense set.

□

Chapter 4

$\sigma_2 DF_n$ and the Definability of Forcing

The simplest approach to defining truth-generic forcing is to show that the second-order definition is really a first-order definition in disguise, or at least that there is a first-order definition equivalent to it. Such techniques are common in set theory: consider, for example, the myriad ways in which large cardinals can be formalized. The problem in the present case is that there seems to be no compelling "reduction" of the second-order recursion defining F to a first-order property

In order to circumvent this difficulty, we will define a new collection of complexity classes – the so-called $\sigma_2 DF_n$ of the chapter title – and show that these complexity classes have nice properties. We will then show that these properties can be used to directly define forcing, provided the partial orders in question meet certain criteria.

4.1 Basics of hyperrecursion and distinguished predicates

Before we can reduce the second-order "hyperrecursion" to a first-order recursion, some conventions must be established to formalize the relevant notions.

The first, and most important, distinction to be drawn is the following:

Definition 4.1. *A* formula *is a syntactic object as per usual. A* predicate *over a model V is a formula using parameters from V; that is, it is a relational symbol to be interpreted by a formula using parameters over V. A* (definable) class *is the instantiation of a predicate over the model, i.e. the actual subset of V determined by the predicate.*

Recursion is an operation on sets, formalized via a class function $F : o(V) \to V$ that recurses over a global class function $G : V \to V$. Similarly, hyperrecursion is an operational on classes, which will (hyper)recurse over a hyperclass function.

Definition 4.2. *A* predicate functional *(hereafter simply* functional*) Ω over a model V is a second-order predicate $\Omega[R]$ in a single second-order variable with parameters from V and no second-order quantifiers.*

Remark 4.3. *Strictly speaking this should be a Δ^1_0 predicate functional, but we shall have no need of higher order functionals.*

Definition 4.4. *Given a functional Ω and a (first-order) predicate φ, we define $\Omega[\varphi]$ to be the result of replacing all occurences of R with φ in Ω. The free variables of $\Omega[\varphi]$ consist of the (first-order) free variables of Ω and the free variables of φ that are not bound by Ω.*

Note that $\Omega[\varphi]$ will be a first-order predicate itself since there are no second-order quantifiers in Ω. We can thus regard Ω as a map from definable classes to definable classes, i.e. $\Omega : \mathcal{D}(V) \to \mathcal{D}(V)$, and in this way Ω becomes the hyperrecursion analog to G above.

Typical examples of such functionals are:

$$\begin{aligned} \Omega[\varphi(\cdot)](x) &= \neg\varphi(x) \\ \Omega[\varphi(\cdot,\cdot)](x,y) &= \forall z \in x[\varphi(z,y)] \\ \Omega[\varphi(\cdot,\cdot,\cdot)](x,y) &= \forall z \exists w \in z[\varphi(z,x,w) \vee \varphi(w,y,w)] \end{aligned}$$

Note that in the latter case that φ is a ternary predicate, but $\Omega[\varphi]$ is binary. In general, we will be flexible about the arities involved; for example, $\Omega[\varphi] = \neg\varphi$ will be considered to be a single functional rather than a collection of functionals for each arity.

To begin, note the following obvious lemma which will be assumed implicitly throughout:

Lemma 4.5. *If $\varphi \leftrightarrow \psi$, then $\Omega[\varphi] \leftrightarrow \Omega[\psi]$.*

Definition 4.6. *Given a well-founded relation E, say that Ω is E-*reduceable *if, given any predicate A and parameters $\overline{a}$, $\Omega[A](\overline{a})$ only requires evaluating A on E-predecessors of $\overline{a}$. If the predicates have a distinguished first tuple (see below), we require that Ω E-reduces the first non-distinguished coordinate when leaving the coordinates fixed.*

Typical examples would be

$$\begin{aligned} aEb \Leftrightarrow a \in b \qquad & \Omega[A](x) \equiv \forall y{\in}x\, A(y) \\ (a,b)E(c,d) \Leftrightarrow (a \in c \wedge b \in d) \qquad & \Omega[A](x,y) \equiv \exists z{\in}x\, \forall w{\in}y\, [A(z,w)] \end{aligned}$$

Remark 4.7. *There's nothing special about the choice of "first non-distinguished coordinate", it's purely for definiteness' sake.*

Finally, a simple definition:

Definition 4.8. *Given a standard coding for formulas – which might be the general coding for Σ_ω or might be specific to the particular class involved – we say that Ω is* computable *iff given an index e for φ, the index e' for $\Omega[\varphi]$ is computable.*

Note that this is a restriction on codings of *formulas*, not on predicates (which are allowed to have set parameters). As such, the usual set-theoretic requirement that such codings be Δ_0 has been altered to the more syntactic notion of computability.

4.2 Definability and hyperclass recursion

Throughout, let $\mathcal{C}$ be a collection of predicates. Typical examples of such $\mathcal{C}$ would be Σ_n, Π_n, or the soon-to-be-introduced $\sigma_2 DF_n$.

Definition 4.9. *Given a collection $\mathcal{C}$, we say that:*

1. $\mathcal{C}$ admits pairing *if any formula $\varphi(\overline{x}, \overline{u}) \in \mathcal{C}$ is (computably) equivalent to some $\psi(\overline{x}, u) \in \mathcal{C}$.*

2. $\mathcal{C}$ admits parameters *if any, for any $a \in V$ and predicate $\varphi(\overline{x}, u_0, \overline{u})$, $\varphi(\overline{x}, a, \overline{u}) \in \mathcal{C}$.*

3. $\mathcal{C}$ *is* stable *if it admits both pairs and parameters.*

The point about stability is that the usual sorts of coding tricks apply, so that we can relate predicates of nominally different arities.

Definition 4.10. *A formula φ* defines *a formula ψ if there is some parameter (aka* code*) e such that $\forall \overline{x}[\psi(\overline{x}) \leftrightarrow \varphi(e, \overline{x})]$. We say that φ defines a collection of predicates iff φ defines each predicate in the collection. Finally, we say that a collection $\mathcal{C}$ is* self-defining *if there is a $\chi_{\mathcal{C}} \in \mathcal{C}$ which defines $\mathcal{C}$.*

This is the usual notion of universal from Σ_n and Π_n. Such a requirement is too restrictive for our present purposes, however, so we will weaken what it means to "define" something.

Definition 4.11. *Let $\psi(\overline{v}, \overline{x})$ be a formula with distinguished first tuple. We say that $\varphi(t, \overline{v}', \overline{x})$* weakly defines *$\psi$ iff there is a code s such that, for every $\overline{a}$ there is a $\overline{a}'$ such that $\forall \overline{x}[\psi(\overline{a}, \overline{x}) \leftrightarrow \varphi(\overline{a}', s, \overline{x})]$. In slightly more technical format, φ weakly defines ψ iff*

$$\exists s \forall \overline{a} \exists \overline{a}' \forall \overline{x}[\psi(\overline{a}, \overline{x}) \leftrightarrow \varphi(\overline{a}', s, \overline{x})]$$

As before, we say that φ weakly defines a collection iff it weakly defines each member of that collection.

Such formulas and predicates will be called *distinguished* when we need to speak of them. The distinguished parameters of any collection of distinguished predicates will be assumed to have the same arity unless specifically noted otherwise.

Definition 4.12. *We say that a collection $\mathcal{C}$ is* weakly closed *under a functional Ω if there is some $\chi_\Omega \in \mathcal{C}$ such that χ_Ω weakly defines $\Omega[\mathcal{C}]$. More explicitly, $\mathcal{C}$ is weakly closed under Ω if there is some $\chi_\Omega \in \mathcal{C}$ such that, for any $\varphi \in \mathcal{C}$ there is a code s such that: given $\overline{a}$ there is some $\overline{a}'$ such that $\Omega[\varphi](\overline{a}, \overline{u}) \leftrightarrow \chi_{\mathcal{C}}(\overline{a}', s, \overline{u})$.*

The key distinction here is that the map $\overline{a} \mapsto \overline{a}'$ need not be definable within $\mathcal{C}$. This map is enormously important, however:

Definition 4.13. *If φ weakly defines ψ, then any map $H(\overline{a}) = \overline{a}'$ producing witnesses to the weak definability is said to be a* weak map *of the definition. The complexity of a weak map is by definition the complexity of H regarding s as a (constant) parameter. If*

φ weakly defines some class $\mathcal{C}$ the weak map must be uniform in the parameter s, i.e. H must satisfy $H(s, \overline{a}) = \overline{a}'$ for all s. If H witnesses $\mathcal{C}$ being weakly closed under Ω, we will sometimes refer to it as the weak closure *of Ω.*

Note that a weak map will in general not be unique, hence the "weakness" of the map: it only gets one possible witness, not all.

Weak closures can be composed:

Proposition 4.14. *Suppose that $\mathcal{C}$ is a stable collection of distinguished predicates, and is weakly closed under two functionals Ω_0 and Ω_1. Then $\mathcal{C}$ is weakly closed under $\Omega_1\Omega_0$, and hence under arbitrary finite compositions of functionals.*

Proof. We turn to the definition: let $\varphi \in \mathcal{C}$ and consider $\Omega_0[\varphi]$. Because $\mathcal{C}$ is weakly closed under Ω_0, there is some χ_{Ω_0} and some parameter s depending only on φ such that

$$\Omega_0[\varphi](a, b, \overline{u}) \Leftrightarrow \chi_{\Omega_0}(a', b', s, \overline{u}) \equiv \chi_{\Omega_0}(a', b', \overline{u}')$$

where $(a', b') = H_0(s, (a, b))$ and we've temporarily added s to the parameters $\overline{u}$ to get $\overline{u}'$. Consider now $\Omega_1\Omega_0[\varphi]$. By the lemma, $\Omega_1[\Omega_0[\varphi](a, b, \overline{u})] \Leftrightarrow \Omega_1[\chi_{\Omega_0}(a', b', \overline{u}')]$. Since $\mathcal{C}$ is also weakly closed under Ω_1, there is a χ_{Ω_1} and a t depending only on that predicate such that, for any a, b

$$\Omega_1\Omega_0[\varphi](a, b, \overline{u}) \Leftrightarrow \chi_{\Omega_1}(a'', b'', t, \overline{u}') \equiv \chi_{\Omega_1}(a'', b'', t, s, \overline{u})$$

where $(a'', b'') = H_1(t, H(s, (a, b)))$. □

Finally, the crux of the matter:

Definition 4.15. *Let $\mathcal{C}$ be a class of distinguished predicates. $\mathcal{C}$ is* weakly self-defining *iff there is a $\chi_{\mathcal{C}} \in \mathcal{C}$ which weakly defines $\mathcal{C}$, and* robust *iff it is stable and weakly-self-defining.*

Remark 4.16. *We require that the same* $\chi_{\mathcal{C}}$ *works for every formula in* $\mathcal{C}$*, including ones whose non-distinguished parameters have different arity than* $\chi_{\mathcal{C}}$*. This is usually accomplished by some kind of coding trick, e.g.* $\varphi(a, u_0, u_1, u_2) \leftrightarrow \chi_{\mathcal{C}}(a', s, u)$ *where* $u = (u_0, u_1, u_2)$*, which in turn points to the need for stability.*

The key point which distinguishes a weakly self-defining class $\mathcal{C}$ from a self-defining one is that the weak map does not itself have to be in $\mathcal{C}$ – or rather, that the composition of the weak map with the predicates doesn't have to be in $\mathcal{C}$.

The central theorem of this section can now be stated:

Theorem 4.17 (Weak Recursion)**.** *Suppose that* $\mathcal{C}$ *is robust. Further suppose that* $\mathcal{C}$ *is weakly closed under an* E*-reduceable functional* Ω*, where* E *is a well-founded relation. Then there is a unique, definable class function* $F : V \to V$ *such that:*

$$\chi_{\mathcal{C}}(F(x), \cdot) \equiv \Omega[\chi_{\mathcal{C}}(F \restriction \mathrm{pred}_E(x), \cdot)]$$

In less complicated language, this says that any collection of classes defined by hyperrecursion with Ω are in fact first-order definable, with $\mathcal{C}$-code given by $F(x)$.

Proof. This is completely straightforward: since $\mathcal{C}$ is weakly closed under Ω, we may apply the regular well-founded recursion theorem to $\mathcal{C}$-codes for predicates, producing the desired result. □

4.3 DF_n and $\sigma_2 DF_n$

In order to apply the Weak Recursion Theorem, we will need to find robust collections that are weakly closed under interesting functionals. To that end, begin by fixing $n \geq 1$, so there is a *universal truth predicate* for Σ_n formulas $\Phi_n \in \Sigma_n$ and a universal truth predicate for Π_n formulas $\Psi_n \in \Pi_n$. We let DF_n be the class of all predicates which are a disjunctive form of Σ_n and Π_n formulas, i.e. $\varphi \in DF_n \leftrightarrow \varphi = \bigvee(\bigwedge \psi_i)$ where each ψ_i is either Σ_n or Π_n. Note that this is exactly like the conventional disjunctive normal form, only we do not require that each predicate appear in every conjunction.

Lemma 4.18. *DF_n is computably closed under negation. That is, given a code e for a DF_n formula φ, there is a (computable) map $e \mapsto e'$ where e' is the code of a DF_n predicate φ' such that $\varphi' \leftrightarrow \neg\varphi$.*

Proof. Apply De Morgan's laws ad libitum and the fact that Σ_n and Π_n are complements of one another. □

Lemma 4.19. *DF_n is computably closed under boolean combinations.*

Proof. Apply the usual results on disjunctive normal form, noting only that the normality is not required here. □

Lemma 4.20. *Σ_n and Π_n are computably closed under Δ_0 functionals. Hence in particular, coding and decoding of ordered pairs, functional evaluation, and the like can be computably absorbed into a Σ_n or Π_n predicate.*

Proof. The syntactic manipulations of the closure of Σ_n and Π_n under Δ_0 operations are effective, although the proof of their correctness requires Σ_n Replacement. □

One of the most important techniques in dealing with DF_n and the related classes is "adjusting", which allows us to modify an existing predicate while preserving its DF_n-ness.

Lemma 4.21. *If φ is a DF_n formula and Q_i are a series of bounded quantifiers, then $Q_0Q_1 \dots Q_k\varphi$ is equivalent to $Q'_0Q'_1 \dots Q'_{k'}\varphi'$ where φ' is also DF_n and the Q'_i are a series of bounded, strictly alternating quantifiers. In other words, bounded quantifiers in front of a DF_N formula can be collapsed without changing its complexity.*

Proof. We illustrate with a simple example, $\exists x\exists y\forall z[\varphi(x,y,z)\vee\psi(x,y,z)]$. This is equivalent to

$$\exists x'\forall z\Big[\exists x \in T(x')\exists y \in T(x')[x' = (x,y) \wedge \varphi(x,y,z)] \vee \\ \exists x \in T(x')\exists y \in T(x')[x' = (x,y) \vee \psi(x,y,z)]\Big]$$

because the x' is common to both clauses, which is (after computably absorbing the external bounded quantifiers) DF_n. The converse is obvious, assuming the decoding is explicitly (and uniformly) contained in each clause. □

We now come to the central definition of this chapter:

Definition 4.22. *φ is σ_2DF_n iff $\varphi(v_1, v_2, \overline{u}) = \exists x{\in}v_1\, \forall y{\in}v_2\, [\widehat{\varphi}(x, y, \overline{u})]$ where $\widehat{\varphi} \in DF_n$. Given a predicate $\varphi \in \sigma_2DF_n$, we will always let $\widehat{\varphi}$ be the DF_n portion of φ.*

Remark 4.23. *There is a very subtle point worth mentioning here: considered as a formula, the* variables *v_1 and v_2 can only occur in the distinguished tuples at the beginning. Considered as a predicate, the* parameters *in those positions may, of course, be utilized in the bound DF_n predicate.*

Proposition 4.24. *There is a formula $\tau \in \sigma_2DF_n$ which weakly defines DF_n, i.e. given any DF_n formula $\varphi(\overline{u})$ there is a code s depending only on $\widehat{\varphi}$ and an a, b such that $\forall\overline{u}[\varphi(\overline{u}) \leftrightarrow \tau(a, b, s, \overline{u})]$.*

Proof. The basic idea for the universal predicate τ is $\exists x{\in}\mathrm{ran}\, s\, \forall e{\in}\mathrm{ran}\, x\, [(e(0) = 0 \wedge \Phi_n(e(1), \overline{p}) \vee (e(0) = 1 \wedge \Psi_n(e(1), \overline{p})]$ where s is the usual code for φ. By adjusting the formulas being used we can see that

$$\begin{aligned}\tau(v_1, v_2, \overline{u}) &= \exists x{\in}v_1\, \forall y{\in}v_2 \Big[y \in \mathrm{ran}(x) \rightarrow [y(0) = 0 \wedge \Phi_n(y(1), \overline{u}) \vee \\ &\qquad y(0) = 1 \wedge \Psi_n(y(1), \overline{u})]\Big] \\ &= \exists x{\in}v_1\, \forall y{\in}v_2 \Big[y \notin \mathrm{ran}(x) \vee [y(0) = 0 \wedge \Phi_n(y(1), \overline{u})] \vee \\ &\qquad [y(0) = 1 \wedge \Psi_n(y(1), \overline{u})]\Big]\end{aligned}$$

satisfies the requirements with $a = \mathrm{ran}(s)$ and $b = T(s)$. □

Remark 4.25. *In future we will take all trivial rearrangements of the DF_n formulas as read, such as the second version of τ given above. Non-trivial rearrangements will, of course, be mentioned explicitly.*

Lemma 4.26. *Assuming AC, $\forall x{\in}a\, \exists y{\in}b\, \varphi(x, y, \overline{u}) \leftrightarrow \exists f{\in}{}^a b\, \forall x{\in}a\, \varphi(x, f(x), \overline{u})$ for any predicate φ.*

Proof. The right-to left implication is obvious: set $y = f(x)$. For the reverse implication, use AC to produce a choice function for the given predicate. □

Theorem 4.27. *$\sigma_2 DF_n$ is weakly self-defining, where the weak map is Π_2, and the parameter s depends only on $\widehat{\varphi}$.*

Proof. Suppose $\varphi(a, b, \overline{u})$ is given by $\exists w{\in}a \forall z{\in}b[\widehat{\varphi}(w, z, \overline{u})]$. We can code $\widehat{\varphi}$ as $\tau(\mathrm{ran}(s), T(s), \overline{u}) = \exists x{\in}\mathrm{ran}(s)\, \forall y{\in}T(s)\, [\widehat{\tau}(x, y, s, w, z, \overline{u})]$ for some code s as above. This gives us

$$\varphi(a, b, \overline{u}) \leftrightarrow \exists z{\in}a\, \forall w{\in}b\, \exists x{\in}\mathrm{ran}(s)\, \forall y{\in}T(s)\, [\widehat{\tau}(x, y, s, w, z, \overline{u})]$$

and since s depends only on $\widehat{\varphi}$ we may move its existential quantifier to the front of φ (and hence disregard it)[1]

$$\varphi(a,b,\overline{u}) \leftrightarrow \exists z{\in}a\, \exists f{\in}^{\mathrm{ran}(s)}b\, \forall w{\in}b\, \forall y{\in}T(s)\, [\widehat{\tau}(f(w),y,s,w,z,\overline{u})]$$

which in turn gives us

$$\varphi(a,b,\overline{u}) \leftrightarrow \exists x{\in}a \sqcup {}^{\mathrm{ran}(s)}b\, \forall y{\in}b \sqcup T(s)\, [\widehat{\Gamma}(x,y,s,\overline{u})]$$

where $\widehat{\Gamma}$ is a suitably adjusted $\widehat{\tau}$. Hence

$$\Gamma(v_1,v_2,\overline{u}) = \exists x{\in}v_1\, \exists y{\in}v_2\, [\widehat{\Gamma}(x,y,t,\overline{u})]$$

weakly self-defines $\sigma_2 DF_n$ with weak map $H(s,(a,b)) = (a \sqcup {}^{\mathrm{ran}(s)}b, b \sqcup T(s))$, which is Σ_3. □

Theorem 4.28. *$\sigma_2 DF_n$ is weakly closed under negation, with weak closure Π_2.*

Proof. $\neg\exists x{\in}a\, \forall y{\in}b\, \varphi \Leftrightarrow \forall x{\in}a\, \exists y{\in}b\, \neg\varphi \Leftrightarrow \exists f{\in}^{a}b\, \forall x{\in}a\, \neg\varphi$, which is in $\sigma_2 DF_n$ because DF_n is closed under negation. □

We can introduce other complexity classes of a similar nature:

Definition 4.29. *For any $m,n \geq 1$, define the class of $\sigma_m DF_n$ predicates to be those obtained by prepending bounded Σ_m quantifiers in front of a DF_n sentence.*

But they gain no real strength:

Proposition 4.30. *For any $n \geq 1$ and $m \geq 2$, $\sigma_2 DF_n$ weakly defines $\sigma_m DF_n$.*

[1] If you want a more formal justification of this, regard s and the parameters $\overline{c}$ substituted for $\overline{u}$ as actual constant symbols added to the language. Since s depends only on $\widehat{\varphi}$, which never changes, we can therefore pull their quantifications outside any series of quantifiers or predicates.

Proof. Iterate Lemmas 4.21 and 4.26 to reverse, then collapse, the bounded quantifiers. □

Having introduced these classes, we must put them into the appropriate place in the usual complexity hierarchy.

Theorem 4.31. *For any $n \geq 1$, the following relations hold:*

$$\Sigma_n, \Pi_n \subsetneq \mathrm{Bool}_n \equiv DF_n \subsetneq \sigma_2 DF_n \subsetneq \Delta_{n+1}$$

Proof. Only the last relation is non-trivial. First, satisfaction for DF_n is definable in Δ_{n+1} since Σ_n and Π_n are uniformized in Δ_{n+1} and boolean combinations thereof can be defined in Σ_{n+1}. Since DF_n is closed under negations via a computable map, this means that $\neg DF_n = DF_n$ is uniformized in $\neg\Sigma_{n+1} = \Pi_{n+1}$ and hence DF_n is uniformized in Δ_{n+1}. $\sigma_m DF_n$ (for any m) is thus uniformized in Δ_{n+1} because Δ_{n+1} is closed under Δ_0 quantifiers, i.e. $Sat_{\sigma_m DF_n} \in \Delta_{n+1}$. But $\sigma_2 DF_n$ is not self-defining, proving the theorem. □

Theorem 4.32. *$\sigma_2 DF_n$ is weakly closed under boolean combinations, where the weak map is Σ_1.*

Proof. Consider $\exists x{\in}a\, \forall y{\in}b\, \varphi(x,y,u) \wedge \exists z{\in}c\, \forall w{\in}d\, \psi(z,w,u)$. This is equivalent to:

$$\exists x'{\in}a'\, \forall y'{\in}b' \Big[\exists x{\in}T(a')\, \forall y'{\in}T(b')\, (\langle 0,x\rangle \in x' \wedge \langle 0,y\rangle \in y' \to \varphi(x,y,u)) \wedge \\ \exists z{\in}T(a')\, \forall w'{\in}T(b')\, (\langle 1,z\rangle \in x' \wedge \langle 1,w\rangle \in y' \to \varphi(x,y,u))\Big]$$

□

and the formula inside the brackets is DF_n. Also, $a' = a \sqcup c$ and $b' = b \sqcup d$, so the weak closure map is Σ_1.

In summary, then:

Theorem 4.33. *The class $\sigma_2 DF_n$ is robust and weakly closed under*

1. *Conjunction and disjunction*

2. *Negation*

3. *Bounded quantification*

Hence, by the Weak Recursion Theorem, any well-founded hyperclass recursion using only negation and set-many unions and interchapters over predicates of fixed complexity is first-order definable.

Note the crucial improvement on the usual Recursion Theorem: one typically recurses over either Σ_n or Π_n functions, but never both. Here, one can recurse over arbitrary combinations of Σ_n and Π_n functions, including bounded quantifiers over these combinations. This will allow much greater latitude in defining classes, as we shall see.

4.4 Regular Complements and $\sigma_2 DF_n$

Fix now a separative partial order $\mathbb{P}$, amenable over some ctm V. Since we are working in a partial order, we will primarily be considering elements of DF_n and $\sigma_2 DF_n$ as classes, i.e. the actual subsets of $\mathbb{P}$ that the predicate determines. Looking at the basic definition of truth-generic forcing, Definition 2.2, note that almost every operation is an operation under which $\sigma_2 DF_n$ is weakly closed (viz. Theorem 4.33), saving only the (order-theoretic) complement $\perp$.

Regarding $\perp$ as a functional $\Omega[A] = A^{\perp}$, the key point about its definition is (abusing notation) that $\Omega[A] \in \Pi_1 \neg(A)$. This means that $\Omega[\Sigma_n] \subset \Pi_n$, while $\Omega[\Pi_n] \subset \Omega[\Pi_{n+1}]$. The trick, however, comes from the basic combinatorial lemmas of the previous chapter: if our classes happen to be regular open, then the complement of a Σ_n predicate is Π_n, while the complement *of a complemented regular open* Σ_n *predicate* is itself Σ_n once more. In other words, instead of repeatedly complementing, we need complement only once and remove complements only once, thus keeping the complexity fixed. The problem is that $\perp$ does not distribute properly over Boolean combinations (viz. Lemma 2.23); accordingly, we will define something slightly different than the usual forcing function F, but which (under a mild restriction on the partial order) will produce the same forcing relation.

To begin:

Definition 4.34. *Define the class of* good $\sigma_2 DF_n$ classes *to be all* $\sigma_2 DF_n$ *classes* φ *with the following restrictions:*

1. *All* Σ_n *and* Π_n *classes in* $\widehat{\varphi}$ *are regular open*

2. *All* Π_n *predicates are given (effectively) as the complement of a* Σ_n *predicate.*

The latter condition could be met by, for example, requiring that the codes for the Σ_n predicates are given as $(+, e)$, where e is a Σ_n code, and Π_n predicates are given as $(-, e)$ where e is the code for the Σ_n complement, changing the uniformizing Π_n predicate accordingly. As none of these changes invalidate the earlier results, we will leave the precise details to the reader and assume that they have been made. Furthermore, by redoing the proofs from the earlier chapters using good predicates, we have:

Theorem 4.35. *All the results from Theorem 4.33 apply to the good $\sigma_2 DF_n$ predicates: it is a robust class that is weakly closed under boolean combinations and bounded quantification.*

In order to make the following definition easier to read, note that good $\sigma_2 DF_n$ classes can be rewritten in the following way:

$$A = \bigcup_{x\in a} \bigcap_{y\in b} \bigcup_{i} \bigcap_{j} A_{ij}$$

where the $A_i j$ are regular open $\Sigma_n \backslash \Pi_n$ classes determined by x, y and whatever other parameters are floating around.

Definition 4.36. *Given a good $\sigma_2 DF_n$ class A, define*

$$A^\dagger = \bigcap_{x\in a} \bigcup_{y\in b} \bigcap_{i} \bigcup_{j} A_{ij}^\perp$$

The definition for predicates is given in the analogous manner.

Put cutely, $A^\dagger$ is what you get if you forget that $\perp$ doesn't distribute over unions.

Proposition 4.37. *The good $\sigma_2 DF_n$ predicates are weakly closed under $\dagger$.*

Proof. The bounded quantifiers may be reversed by Lemma 4.26 as usual; the change from conjunctive form to disjunctive form also works as usual; and the fact that the predicate is good means that $A_{ij}^\perp$ can be coded as either a Π_n predicate (by noting that it is the complement of the original Σ_n predicate) or a Σ_n predicate (by removing the complement signal). □

The key combinatorial lemma is the following:

Proposition 4.38. *Suppose $\mathbb{P}$ is κ-closed for every κ, that A is a good $\sigma_2 DF_n$ class and that $A \subset^* B$. Then $A^\dagger \subset^* B^\perp$.*

Proof. To begin with, note that we need only show that $A^\dagger \subset^* A^\perp$ since, by Lemma 2.23, $A^\perp = B^\perp$. But this follows immediately from Lemmas 2.22 and 2.23, using the fact that $\mathbb{P}$ is sufficiently closed. □

We can now define a new function G which is exactly like the original "strong forcing function" F from Definition 2.2, except that $\perp$ is replaced by $\dagger$ in the quantifer-free clauses.

Proposition 4.39. *G is a uniformly definable function over $(V;\mathbb{P})$.*

Proof. Fix any $n \geq 1$. The definition of G uses boolean combinations, bounded quantifiers and $\dagger$, under which the good $\sigma_2 DF_n$ classes are weakly closed. The atomic classes used in the construction are of the form $[p]$ which are regular open and of low complexity, so the base cases are all good $\sigma_2 DF_n$. Well-founded weak recursion over $\sigma_2 DF_n$ thus shows that $G^\pm(\varphi)$ is definable for quantifier free sentences, and hence G is definable in general. □

Finally, putting all the pieces together, we have:

Theorem 4.40. *Suppose that $\mathbb{P}$ is a separative, amenable partial order and that $\mathbb{P}$ is κ-closed for every κ. Then truth-generic forcing over $\mathbb{P}$ is definable.*

Proof. Note that $G^\pm(\varphi) \subset^* F^\pm(\varphi)$ for quantifier-free φ by repeated applications of Proposition 4.38 and by Lemma 2.25, so we have $p \Vdash \varphi$ iff $F^+(\varphi)$ is dense below p iff $G^+(\varphi)$ is dense below p. But $G^+(\varphi)$ is definable by Proposition 4.39, so forcing is thus definable. □

Chapter 5

Conclusion

5.1 Further work

Although many results concerning truth-generic forcing have been proven in this paper, there is still much work to be done before its theory is as elegant as that of ordinary forcing. Below, I list several areas where further research should prove fruitful.

5.1.1 Intuitionistic Logic

One of the most intriguing aspects of truth-generic forcing is the way in which intuitionistic logic is buried almost definitionally inside: $p \Vdash \varphi$ iff $p \in F^+(\neg\neg\varphi) \supset F^+(\varphi)$. It is not true that F^+, the "strong forcing relation" obeys intuitionistic logic as a general rule, but there certainly appear to be some connections that one might tease out with sufficient delicacy, the prototype being Proposition 2.31. The advantage of this approach is apparent from some of the early theorems in Chapter 2: by removing the need to repeat basic steps ad nauseum, which is not only tedious but can obscure that which should be clear, it allows one to focus on the meatier aspects of the argument.

5.1.2 Pretameness and tameness for truth-genericity

Pretameness, as a concept, is a very elegant. One simply needs that dense classes reduce to predense sets in a suitably nice way and not only does this suffice for useful forcing arguments like the porting of generics, it is also equivalent to preserving ZFC. It is almost the platonic ideal of "the right notion".

Truth genericity seems to lack this elegance, though of course a belief in its absence does not constitute a proof. The central problem is that, while density is a combinatorial condition placed directly on the other, the classes $F(\varphi)$ are not intuitive. For example, we do not yet have a good characterization of what classes which are dense, but not a $F(\varphi)$ for any φ, are like. This becomes doubly complex when, as in the definition of pretameness, one needs a dense collection of dense things; there are limits to the self-referential formulas we can understand!

Nevertheless, there are reasons to believe that one should be able to extract from the crude formula-inspired mechanics a more elegant, combinatorial approach to truth genericity. In addition to a model-theoretic approach as per Woodin's argument in [6], wherein one simply adds stratified satisfaction predicates to the language, there are possibilities involving more direct reflection arguments (e.g. using the indiscernibles) to try to capture, at the set level, some of what is happening at the class level.

5.1.3 The splitting of completions

Every partial order $\mathbb{P}$ embeds into its set-completion. Analogous to other forcing arguments, there should be a good description of those orders whose forcings are equivalent to their completions – good, if not optimal – but again the "right" description seems elusive. Theorem 3.7 is a good start, and represents one of the oddities of the boolean algebraic approach: since it is the symmetries of the algebra that allow forcing to be definable, the hypotheses of such embedding theorems will necessarily have a symmetric flavor. To be more precise, it is the existence of complements in the algebra which makes forcing definable, as it allows the $\perp$ operator to act on elements rather than on classes, but complementation reverses the order on the algebra; therefore any attempt at controlling the algebra must control it both in the original order and the opposite order. Whether this produces something interesting or useful in forcing applications – which are exclusively concerned about downward extensions – is something that has yet to be determined.

5.1.4 Intermediate complexity classes

Having defined the classes $\sigma_m DF_n$, it is natural to ask other questions: what are their relationships? Are there any other "natural" complexity classes lurking between Σ_n and Δ_{n+1} and, if so, how are they natural and what can they be used for? While certain relations seem obvious, the subtleties of the bounded quantifiers prevents most of the standard techniques from applying.

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