Effectivizations of Dimension and Cardinal Characteristics

By

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A dissertation submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

(Mathematics)

at the

UNIVERSITY OF WISCONSIN – MADISON

2017

Date of final oral examination: May 4, 2017

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Abstract

This thesis has two distinct parts. Both involve the effectivization of ideas from other areas of mathematics, one from geometric analysis and one from set theory.

Chapter 1 will be devoted to notions of dimension. One definition of Hausdorff dimension can be given in terms of null sets for Hausdorff measure. Altering this definition by replacing null sets with effective null sets gives one of many characterizations of effective Hausdorff dimension. We look at what happens when we apply an alternate criterion for effectively null. This gives us a new dimension notion which is unique from any previously studied. We prove some of the properties of this new definition.

Chapter 2 will be spent exploring the translation scheme for converting cardinal characteristics to computability-theoretic highness notions as first studied by Rupprecht. We will give the definitions for and characterize the effectivizations of four different cardinal characteristics.

Acknowledgements

First, I'd like to thank the entire logic group at UW-Madison for their help and support. From the moment that I expressed any interest in doing logic, I've felt supported. Even though the size of the group means that all the graduate students are used to being voluntold for tasks, it also means I've always felt included. I'm thankful to have taken classes and learned from every member of the faculty and with most of the other graduate students. Thanks also to Ivan, my coauthor for chapter 2, without whom I would have had a hard time making either heads or tails of some exotic forcings.

Thanks also to my family. I'd like to thank my father for helping to keep in perspective the task of writing a doctoral thesis, my mom for being stoically supportive, and both of them for regular visits featuring beer and zucchini bread from home. Thanks to my brother for keeping me grounded and making me laugh and my sister for the regular visits and being the easiest houseguest imaginable.

Thanks to friends including, but not limited to, Diane, Megan, Quinton, Avery, Kyle, and Jen for many a delightful evening spent around our table sharing food, fellowship, and maybe a drink or two.

Thanks to my advisor Joe Miller for always being supportive and helpful. Joe has always been patient in explaining a concept, even if it's what seems like the tenth time.

Finally, thanks to my wife Emily for being generally amazing. I'm pretty sure that for 90% of the last 6 years, she's believed in me more than I have myself. Coming to Madison together has involved some of the best (and worst) times in our lives together, and I'm glad to have always had someone to share that adventure with.

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1.1 Introduction

We will give here a short introduction to the idea of effective dimension including some of its properties and history. For a more thorough treatment, see [28].

Hausdorff dimension can be thought of as a refinement of notions of measure, specifically as a refinement of the notion of null sets. In one dimension, which is what we will focus on in this thesis, it provides an infinite gradation for the null sets. The history of interplay between measure theory and computability theory is quite old, dating at least to the work of Martin-Löf [24] on random sequences. The relationship between Hausdorff dimension and computability was first introduced much later, in 2000 by Lutz [21], [22]. We present one version of the classical definition of Hausdorff dimension, and then show how one characterization of effective dimension arises naturally from simply replacing the concepts in the definition with appropriate effective analogs.

Definition 1.1. For $X \subseteq 2^\omega$, we define a *δ-cover of X* as a collection of basic open sets in Cantor space $\{[\sigma_i]\}_{i\in\omega}$ with $X \subseteq \bigcup_{i\in\omega} C_{\sigma_i}$ and for all $i, 2^{-|\sigma_i|} < \delta$. Then we define

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i\in\omega} 2^{-s|\sigma_i|} : \{C_{\sigma_i}\}_{i\in\omega} \text{ is a } \delta\text{-cover of } X \right\}.$$

The *Hausdorff s-measure* of X is then defined by

$$\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(X).$$

Finally, the *Hausdorff dimension* of X is defined by

$$\dim(X) = \inf \{s \in [0,1] : \mathcal{H}^s(X) = 0\}.$$

That is, the infimum of all s for which X is Hausdorff s-null. One of the equivalent characterizations of effective Hausdorff dimension as introduced by Lutz [22] is a direct effective analog of the definition above. By replacing "Hausdorff s-null" in the above statement with "effectively Hausdorff s-null" we recover exactly the definition of effective dimension as follows.

Definition 1.2. A computable sequence of c.e. sets of finite strings $(S_n)_{n\in\mathbb{N}}$ is an *s-test* if it satisfies

$$\mathrm{DW}_s(S_n) \leq \frac{1}{2^n}.$$

A class $A \subset 2^\omega$ is *s-null* if there exists a weak 2 s-test (S_n) such that

$$A \subseteq \bigcap_{n\in\omega} [S_n].$$

The *effective Hausdorff dimension* (or just *effective dimension*) of a set X is

$$\dim_{W2}(X) = \inf\{s \geq 0 : \{X\} \text{ is weak 2 } s\text{-null}\}.$$

In computability there are a number of competing notions of "effectively null." The one that is used here is a direct analog of a Martin-Löf test. Work has also been done with other competing notions of effectively null instead. Replacing Martin-Löf randomness with computable randomness can be seen in [22] and [33] or with Schnorr randomness in [9]. These two dimension notions are equivalent, as is shown in[9].

In this thesis, we will define a new dimension notion by changing the type of effective nullity in this definition to one analogous to the weak-2 test, a stronger randomness notion. In section 1.3 we will give this definition and demonstrate that it is distinct from other effective dimension definitions. In section 1.5 we will examine the lowness property associated with our new definition and characterize its relationship with low for dimension as studied in [20].

1.2 Preliminaries

In trying to work with effective versions of these definitions, we need to be more careful about the way that we talk about weights of collections than in the non-effective case. In particular, we will use the following definitions of weight over the course of this chapter.

Following Miller [25] we define the following:

Definition 1.3. Given a c.e. set of finite strings $S \subset 2^{<\omega}$, the *direct s-weight* of S is defined as

$$\mathrm{DW}_s(S) = \sum_{\sigma \in S} 2^{-s|\sigma|}.$$

The *prefix-free s-weight* of S is defined as

$$\mathrm{PW}_s(S) = \sup\{\mathrm{DW}_s(V) : V \subseteq S \text{ is prefix-free}\}$$

The *optimal s-weight* (sometimes called *vehement s-weight*) of S is defined as

$$\mathrm{OW}_s(S) = \inf\{\mathrm{DW}_s(V) : [S] \subseteq [V]\}.$$

We note that, by definition, optimal weight $\leq$ prefix-free weight $\leq$ direct weight. Similarly, for a Σ^0_1 class A, we can define optimal s-weight as

$$\mathrm{OW}_s(A) = \inf\{\mathrm{DW}_s(V) : A \subseteq [V]\}.$$

We further define conditional weight for $\sigma \in 2^{<\omega}$ in two cases. First, if there is some $\tau \prec \sigma$ with $\tau \in S$, we define both optimal and direct weight conditioned on σ to be 1. Otherwise, if S contains no prefixes of σ, then we define the conditional weight by

$$\mathrm{OW}_s(S|\sigma) = \mathrm{OW}_s\{\tau \in 2^{<\omega} : \sigma ^\frown \tau \in S\}.$$

$$\mathrm{DW}_s(S|\sigma) = \mathrm{DW}_s\{\tau \in 2^{<\omega} : \sigma ^\frown \tau \in S\}.$$

It will be convenient to be able to pass relatively freely between these weight notions, and it turns out that we can do so effectively so long as we allow slippage in the dimension

s.

The following definition was also given by Miller in [25].

Definition 1.4. The *optimal cover* of $S \subseteq 2^{<\omega}$ is a set $S^{oc} \subseteq 2^{<\omega}$ such that $[S] \subseteq [S^{oc}]$ and $\mathrm{DW}_s(S^{oc}) = \mathrm{OW}_s(S)$. For the sake of uniqueness, if there are multiple such sets, we define S^{oc} to be the measure-least such set.

We now reproduce a lemma of Miller. This is Lemma 3.3 from [25].

Lemma 1.5. *For any c.e. set $S \subseteq 2^{<\omega}$, we can (effectively) find a c.e. $V \subseteq 2^{<\omega}$ such that $[V] = [S^{oc}]$ and if $P \subseteq V$ is prefix-free, then $DW_s(P) \leq OW_s(S)$.*

See proof in [25].

Lemma 1.6. *For all $s < t$ and every Σ^0_1 class A with $OW_s(A) = a$, there is a c.e. $V \subseteq 2^{<\omega}$ such that $A \subseteq [V]$ and $DW_t(V) \leq a \cdot C(s,t)$ where $C(s,t)$ is a constant depending only on s,t. Moreover, given a c.e. description $S \subseteq 2^{\omega}$ with $A = [S]$, V can be found effectively in S.*

Proof. Let s, t, and A be as in the statement of the lemma. Let $S \subseteq 2^{<\omega}$ be c.e. with $A = [S]$. Applying Lemma 1.5, there is a c.e. $V \subseteq 2^{<\omega}$ such that $A = [S] \subseteq [V]$ and every prefix-free subset $P \subseteq V$ has $\mathrm{OW}_s(P) \leq a$. We will use this fact to get the bound on the direct t-weight of V that we desire. In particular, we note that the collection $V_i = \{\sigma \in V : |\sigma| = i\}$ is prefix-free, and so we can apply the above bound to find that for all i, $\mathrm{DW}_s(V_i) \leq a$. But then $\mathrm{DW}_t(V_i) \leq 2^{(s-t)i}a$ and so we have

$$\begin{aligned}\mathrm{DW}_t(V) &= \mathrm{DW}_t\left(\bigcup_i V_i\right)\\ &= \sum_i \mathrm{DW}_t(V_i)\\ &\leq \sum_i 2^{(s-t)i}a\\ &= \frac{1}{1-2^{s-t}} \cdot a\end{aligned}$$

as desired, with $C(s,t) = \frac{1}{1-2^{s-t}}$. □

It will also be convenient to be able, in certain circumstances, to deal with specific covers with other nice properties.

Definition 1.7. We say that $U \subseteq 2^{<\omega}$ is s-closed if for every $\tau \in 2^{<\omega}$, $\mathrm{OW}_s(U|\tau) \geq 1 \Rightarrow \tau \in U$. Similarly we say that a set $V \subseteq 2^{\omega}$ is s-closed if for every $\tau \in 2^{<\omega}$, $\mathrm{OW}_s(V|\tau) \geq 1 \Rightarrow [\tau] \subseteq V$.

It turns out that we can turn a c.e. sequence of strings into a c.e. s-closed sequence while adding an arbitrarily small amount of optimal weight. (This lemma for prefix-free weight appears in [20]. The construction given is nearly identical, but the proof in the optimal weight case is more involved.)

Lemma 1.8. *Let $\varepsilon > 0$, $s \in (0,1]$ rational, and $S \subseteq 2^{<\omega}$ be a c.e. collection of strings. Then there is a c.e. collection $V \subseteq 2^{<\omega}$ with the following properties*

- $[S] \subseteq [V]$.
- $OW_s(V) \leq OW_s(S) + \varepsilon$.

- *V is s-closed.*

Proof. Assuming, without loss of generality, that ε is rational, let $V \subseteq 2^{<\omega}$ be the closure of S under the requirement that for every $\tau \in 2^{<\omega}$, if there is a finite, prefix-free set $W \subseteq V$ of extensions of τ such that $\mathrm{OW}_s(W) > 2^{-s|\tau|} - 2^{-2|\tau|-1}\varepsilon$, then $\tau \in U$. We note that this process is computable, as the optimal weight of any finite collection of strings is computable. Given this, it is easy to see that V is c.e. and s-closed.

To see that the resultant set V has the optimal weight we claimed, let $\delta > 0$, then there is an $S^* \subseteq 2^{<\omega}$ such that $\mathrm{DW}_s(S^*) < \mathrm{OW}_s(S)+\delta$ and $[S] \subseteq [S^*]$. Then let $\{\tau_n\}_{n\in\omega}$ the collection of all the τ that are added in the closure process described above. We then define V^* as the collection of minimal strings in $\sigma \in S^* \cup \bigcup_{n\in\omega} \tau_n$—that is, strings have no predecessors in the collection. Then V^* clearly has $[V] \subseteq [V^*]$ by construction. Additionally, we claim that $\mathrm{DW}_s(V^*) < \mathrm{OW}_s(S) + \delta + \varepsilon$. To see that this is true, we note that V^* can be viewed as the limit of a process whereby we start with S^* as a cover of S and while constructing V, every time that a τ enters, we add τ to our cover and remove every string with τ as a prefix. Every time we add a τ, because of the way that our construction is structured, we only add direct weight of less than $2^{-2|\tau|-1}\varepsilon$. Further, each τ can be added to V at most once, and so we have

$$\mathrm{OW}_s(V) \leq \mathrm{DW}_s(V^*) < \mathrm{DW}_s(S^*) + \sum_{\tau\in 2^{<\omega}} 2^{-2|\tau|-1}\varepsilon < \mathrm{OW}_s(S) + \delta + \varepsilon.$$

However, $\delta > 0$ was arbitrary, and so we have $\mathrm{OW}_s(V) \leq \mathrm{OW}_s(S) + \varepsilon$, as desired. ∎

1.3 Weak 2 Dimension

Now, we can define weak 2 dimension completely analogously to the covering definition of Hausdorff dimension.

Definition 1.9. A computable sequence of c.e. sets of finite strings $(S_n)_{n\in\mathbb{N}}$ is a *weak 2 s-test* if it satisfies

$$\lim_{n\to\infty} \mathrm{DW}_s(S_n) = 0.$$

A class $A \subset 2^\omega$ is *weak 2 s-null* if there exists a weak 2 s-test (S_n) such that

$$A \subseteq \bigcap_{n\in\omega} [S_n].$$

The *weak 2 Hausdorff dimension* (or just *weak 2 dimension*) of a set X is

$$\dim_{W2}(X) = \inf\{s \geq 0 : \{X\} \text{ is weak 2 } s\text{-null}\}.$$

All definitions relativized to a sequence A are defined analogously.

We note that in the case of both effective Hausdorff dimension and computable/Schnorr dimension there are nice alternative characterizations in terms of both complexity and martingales. Weak 2 dimension does not lend itself to these characterizations as there are no known characterizations of weak 2-randomness via martingales or complexity.

Now, having defined weak 2 dimension, we aim to demonstrate that it truly is different from the other existing effective dimension notions. Weak 2 dimension is clearly no greater than effective dimension, the strongest effective dimension notion studied. We will show that it differs from effective dimension in the strongest possible way. Namely, there are Martin-Löf random sequences (all of which have effective dimension 1) with weak 2 dimension 0.

Theorem 1.10. *If $Z \in 2^\omega$ is Δ^0_2 then for every $s > 0$, $\{Z\}$ is an s-null Π^0_2 class. Hence, Z has weak 2 dimension 0.*

Proof. We adapt the proof that Δ^0_2 sequences are not weak 2-random from [26] with a little bit of extra trickery. Let $s > 0$ and $(Z_n)_{n\in\omega}$ be a computable approximation of Z. We build a c.e. sequence $(V_m)_{m\in\omega}$ such that $\{Z\} = \bigcap_m V_m$. We will then conclude that the V_m as we constructed them have optimal $(s/2)$-weight tending to zero, and hence, as discussed above, there is a c.e. sequence $(W_m)_{m\in\omega}$ such that $\lim_{m\to\infty} \mathrm{DW}_s(W_m) = 0$ and $Z \in \bigcap_m W_m$, as desired.

To enumerate V_m, for each $n > m$, if x is least such that $Z_n(x) \neq Z_{n-1}(x)$, put $[Z_n \restriction x]$ into V_m. Since for all x, $Z_n(x)$ can only change finitely many times, it follows that for all x there is some stage t_x after which $Z_n \restriction x$ is stable. Thus, we have that the optimal $(s/2)$-weight of $V_m \leq 2^{-sx/2}$ for all $m > t_x$. Since x was arbitrary, this means that the optimal $(s/2)$-weight of the V_m vanishes as $m \to \infty$. Finally, we apply Lemma 1.6 to get c.e. (W_m) such that

$$\lim_{m\to\infty} \mathrm{DW}_s(W_m) = 0.$$

So we see that Z is weak 2 s-null. However, since $s > 0$ was arbitrary, it follows that Z has weak 2 dimension 0. □

Corollary 1.11. *There is a Martin-Löf random with weak 2 dimension 0.*

Proof. For an example, take any random which is Δ^0_2, e.g. Chaitin's Ω. □

1.4 Partial Weak 2-Randomness

Another route to achieve an alternative characterization of weak 2 dimension is by way of partial weak 2-randomness.

Definition 1.12. We define $X \in 2^\omega$ to be *weak 2 s-random*, *prefix-free weak 2 s-random*, or *optimal weak 2 s-random* if

$$X \notin \bigcap_{n\in\omega}[A_n]$$

whenever $\{A_n\}$ is a uniformly c.e. sequence with $A_n \subseteq 2^{<\omega}$ and $\lim_{n\to\infty} \mathrm{DW}_s(A_n) = 0$, $\lim_{n\to\infty} \mathrm{PW}_s(A_n) = 0$, or $\lim_{n\to\infty} \mathrm{OW}_s(A_n) = 0$, respectively.

We note that an alternative characterization of the definition of weak 2 dimension can be given in terms of any of these partial randomness notions by

$$\dim_{W2}(X) = \sup\{s \in [0,1] : X \text{ is (prefix-free/optimal) weak 2 } s\text{-random}\}.$$

Theorem 1.13. *For all $0 \leq s < t \leq 1$ weak 2 t-randomness implies prefix-free weak 2 s-randomness.*

Proof. We note that the contrapositive follows directly as a result of the fact that prefix-free weight $\geq$ optimal weight. We can apply Lemma 1.5 to members of our weak 2 s-test to get members of a prefix-free weak 2 t-test with direct weight no more than a fixed constant multiple of the prefix-free weight of the original set. □

Lemma 1.14. *For all $s \in [0,1]$, every optimal weak 2 s-test $\{A_n\}_{n\in\omega}$ is covered by a prefix-free weak 2 s-test.*

Proof. Let $s \in [0,1]$ and a uniformly c.e. sequence $\{A_n\}_{n\in\omega}$ an optimal weak 2 s-test. Then, applying Lemma 1.5 to an enumeration $\{a_{i,n}\}$ with $\bigcup_i [a_{i,n}] = A_n$, we can effectively find a c.e. set $V_n \subseteq 2^{<\omega}$ which is also a cover of A_n and which has that all prefix-free subsets $P \subseteq V_n$ have $\mathrm{DW}_s(P) \leq \mathrm{OW}_s(A_n)$, thus $\mathrm{PW}_s(V_n) \leq \mathrm{OW}_s(A_n)$, and so $\{V_n\}_{n\in\omega}$ is a prefix-free weak 2 s-test covering $\{A_n\}$. □

Theorem 1.15. *For all $s \in [0,1], X \in 2^\omega$, X is prefix-free weak 2 s-random if and only if X is optimal weak 2 s-random.*

Proof. (⇐) This direction is a trivial result of the fact that prefix-free weight ≥ optimal weight, hence any prefix-free weak 2 s-test is already a optimal weak 2 s-test.
(⇒) This is a direct result of Lemma 1.14. □

These are all the implications that we have been able to show. We note that the remaining pair of notions (prefix-free and normal) can be separated in the s-random case, as can be seen in the following theorem. (In fact, if we generalize further the notion of partial randomness, there is a separate notion using a Solovay-type test that falls strictly between the two.)

Theorem 1.16 (Reimann and Stephan, 2006)**.** *For all $s \in (0,1)$, there exists $X \in 2^\omega$ such that X is s-random but not prefix-free s-random.*

For proofs, see [12] and [29].

We would hope to be able to achieve the corresponding separation for weak 2 s-randomness and prefix-free weak 2 s-randomness by building $A \in 2^\omega$ which is weak 2 2-random but not prefix-free weak 2 s-random for some s, but have not yet successfully separated the two. However, we can narrow down the possible computational properties of such an A.

Theorem 1.17. *If X is (prefix-free) s-random and forms a minimal pair with $0'$, then X is (prefix-free) weak 2 s-random.*

Proof. We will prove the case for direct weight, the prefix-free proof is identical. We prove by contradiction. Let X be s-random but not weak 2 s-random. Then there is a nested weak 2 s-test $\{V_n\}$ with $X \in \bigcap_{n\in\omega} [V_n]$. We will simultaneously build a set A which is simple and a Turing operator Γ which witnesses that $X \geq_T A$. We define the requirement

$$R_e : |W_e| = \infty \Rightarrow A \cap W_e \neq \emptyset.$$

Let $A_0 = \emptyset$, Γ_0 empty. We begin to enumerate the $\{V_n\}$ and W_e. At stage $t > 0$, for each $e < t$ if R_e has not been met so far, and we see $n > 2e$ enter W_e and $\mathrm{DW}_s(V_{n,t}) \leq 2^{-e}$, then we define $A_t = A_{t-1} \cup \{n\}$ and say that R_e is met. Additionally, whenever we see σ enter $V_{n,t}$, we define $\Gamma^Y(n) = A_t(n)$ for all $Y \in [\sigma]$.

We claim that A is a simple set. It is clear that if W_e is infinite, then R_e eventually acts, and so $A \cap W_e \neq \emptyset$. Further, A must be coinfinite because the nth element of A is at least $2n$. Additionally, we claim that $\Gamma^X =^* A$ and so $A \leq_T X$. To see this, we note that successively smaller tails of the collection $\{V_{n,t} : n \text{ enters } A \text{ at stage } t\}$ form a Martin-Löf s-test, and so X can be in at most finitely many of the $V_{n,t}$, and so for all but finitely many n, $\Gamma^X(n) = A(n)$. □

Theorem 1.18. *If X is s-random but not prefix-free s-random, then $X \geq_T 0'$.*

Proof. Let $\{A_n\}$ be a prefix-free s-test such that $X \in \bigcap_{n\in\omega} [A_n]$. We will build a uniformly c.e. collection of computable sets of strings $\{B_n\}$. We begin an enumeration of $0'$ while simultaneously enumerating $\{A_n\}$. When we see i enter $0'$, we will add a maximal prefix-free subset of the part of A_i that we have enumerated thus far as B_i. We note that $DW_s(B_n) \leq 2^{-n}$ by definition of our prefix-free s-test. We then let $C_n = \bigcup_{i>n} B_i$. We note that if $X \in [B_n]$ for infinitely many of the B_n, then $X \in [C_n]$ for all $n \in \omega$ and as

$$\mathrm{DW}_s(C_n) = \sum_{i>n} \mathrm{DW}_s(B_i) \leq \sum_{i>n} 2^{-i} = 2^{-n}$$

we have that $\{C_n\}$ is an s-test covering X, but by assumption, X is s-random, and so not contained in any s-test. Then it must be the case that X is contained in at most finitely many of the $[B_n]$.

Thus, we can compute all but finitely many bits of $0'$ (and hence $0'$) by trying to enumerate $0'$ and $\{A_n\}$ simultaneously and concluding that if we see X enter $[A_n]$ before n enters $0'$, then $n \notin 0'$. Thus $X \geq_T 0'$, as desired. □

Corollary 1.19. *If X is weak 2 s-random and forms a minimal pair with $0'$, then X is prefix-free weak 2 s-random.*

Proof. This is a direct result of the above theorems. If X is weak 2 s-random, then S is s-random by definition. Since X forms a minimal pair with $0'$ then, in particular, $X \not\geq_T 0'$, and so by Theorem 1.18 X is prefix-free s-random. Finally, since X forms a minimal pair with $0'$, it follows from Theorem 1.17 that X is prefix-free weak 2 s-random. □

1.5 Lowness Properties

A common question of interest when dealing with effective definitions and their relativizations is to ask about the corresponding lowness notion. In particular, the collection of oracles which are low for a given notion is, roughly speaking, those oracles which have no more computational strength from the perspective of the given notion than the unrelativized version.

Definition 1.20. We say that A is *low for weak 2 dimension* if for all $X \subseteq 2^\omega$

$$\dim_{W2}(X) = \dim_{W2}^A(X).$$

Similarly, A is *low for effective dimension* if for all $X \subseteq 2^\omega$

$$\dim(X) = \dim^A(X).$$

Lempp, Miller, Nies, Turetsky, and Weber in [20] gave many alternate characterizations of low for dimension. The main theorem of the paper providing these equivalences is as follows:

Theorem 1.21. *The following are equivalent:*

(1) *A is lowish for Martin-Löf random—that is, if X is Martin-Löf random, then $\dim^A(X) = 1$.*

(2) *A fixes a single dimension s—that is, there exists $s \in (0, 1]$ such that for all $X \in 2^\omega$ we have that*

$$\dim(X) = s \Rightarrow \dim^A(X) = s.$$

(3) *A is low for effective dimension.*

(4) *A has the Σ^0_1-covering property, i.e., if $W \subseteq 2^{<\omega}$ is A-c.e., $0 \leq s < t \leq 1$, and $DW_s(W) < 1$, then there is a c.e. set V with $DW_t(V) < 1$ such that $[W] \subseteq [V]$.*

(5) *A has the c.e. covering property, i.e., if* $W \subseteq 2^{<\omega}$ *is A-c.e.,* $0 \leq s < t \leq 1$, *and* $DW_s(W) < 1$, *then there is a c.e. set* $V \supseteq W$ *with* $DW_t(V) < \infty$.

(6) $\liminf_{|\sigma|\to\infty} \dfrac{K^A(\sigma) - K(\sigma)}{|\sigma|} \geq 0.$

(7) *A is lowish for K, i.e.* $\liminf_{|\sigma|\to\infty} \dfrac{K^A(\sigma)}{K(\sigma)} \geq 1.$

We have no immediately apparent way to convert (4) and (5) to versions for weak 2 dimension, (6) and (7) are even less likely candidates, as they deal with complexity bounds which have no immediate counterpart in weak 2-randomness or dimension, but (1), (2), and (3) lend themselves nicely to the weak 2 version of dimension. Their counterparts are presented below.

Conjecture 1.22. *The following are equivalent:*

(1′) *A is lowish for weak 2-random—that is if X is weak 2-random, then* $\dim^A_{W2}(X) = 1$.

(2′) *A fixes a single weak 2 dimension, that is, there exists* $s \in (0, 1]$ *such that for all* $X \in 2^\omega$ *we have that*

$$\dim_{W2}(X) = s \Rightarrow \dim^A_{W2}(X) = s.$$

(3′) *A is low for weak 2 dimension.*

While it seems likely that the full equivalence as stated is true, we do not have a complete proof. What we can prove are the following.

We first note that (3′) ⇒ (2′) is trivial.

We will prove a slightly weaker version of (2′) ⇒ (1′), but first we will need the following lemma:

Lemma 1.23. *If s is a computable real and $s[Z] = \{\lfloor n/s \rfloor : Z(n) = 1\}$, then*

$$\dim_{W2}(s[Z]) = s \cdot \dim_{W2}(Z).$$

Proof. To see that $\dim_{W2}(s[Z]) \leq s \cdot \dim_{W2}(Z)$, let $t = \dim_{W2}(Z)$ and $r > t$. Then there exists a weak 2 r-test $\{A_n\}_{n\in\omega}$ c.e. such that $\forall n \in \omega(Z \in [A_n])$ and $A_n = \{a_{i,n} | i \in \omega\}$ with

$$\lim_{n\to\infty} \sum_i 2^{-r|a_{i,n}|} = 0$$

then defining $b_{i,n} = \{s[x] : x \in a_{i,n}\}$ for each $i, n \in \omega$, and defining $B_n = \{b_{i,n} | i \in \omega\}$ for all n, we have that $\forall n \in \omega(s[Z] \in [B_n])$ and

$$\begin{aligned} \lim_{n\to\infty} \sum_i 2^{-rs|b_{i,n}|} &\leq \lim_{n\to\infty} \sum_i 2^{-r(|a_{i,n}|+1)} \\ &= \lim_{n\to\infty} 2^{-r} \sum_i 2^{-r|a_{i,n}|} \\ &= 0 \end{aligned}$$

Thus we have that $\dim_{W2}(s[Z]) \leq sr$, but as $r > t$ was arbitrary, it follows that $\dim_{W2}(s[Z]) \leq st$.

To see that $\dim_{W2}(s[Z]) \geq s \cdot \dim_{W2}(Z)$, let $t = \dim_{W2}(s[Z])$ then for all $r > t$ there exists a weak 2 r-test $\{A_n\}_{n\in\omega}$ c.e. with $A_n = \{a_{i,n} | i \in \omega\}$ such that $s[Z] \in [A_n]$ for all n and

$$\lim_{n\to\infty} \sum_i 2^{-r|a_{i,n}|} = 0.$$

Define $b_{i,n} = \{x \restriction \{\lfloor j/s \rfloor\}_{j\in\omega} | x \in a_{i,n}\}$ (that is we only take bits from places of the form $\lfloor j/s \rfloor$, omitting all others). Then defining $B_n = \{b_{i,n} | i \in \omega\}$ we have $Z \subseteq [B_n]$ for all n

and

$$\lim_{n\to\infty}\sum_i 2^{\frac{-r}{s}|b_{i,n}|} \leq \lim_{n\to\infty}\sum_i 2^{-r|a_{i,n}|} = 0.$$

Thus, $\dim_{W2}(Z) \leq \frac{r}{s}$, but again as $r > t$ was arbitrary, it follows that $\dim_{W2}(Z) \leq \frac{t}{s}$.

These two combined give us the desired equality. □

Using these we will prove partial versions of both $(2') \Rightarrow (1')$ and $(2') \Rightarrow (3')$.

Theorem 1.24. *If A fixes a single computable weak 2 dimension s, then A is lowish for weak 2-random.*

Proof. Let X be weak 2-random and a computable s be as in the statement of $(2')$. Applying Lemma 1.23 twice and $(2')$ once, we can see that:

$$\begin{aligned} s\dim_{W2}(X) &= \dim_{W2}(s[X]) \\ &= \dim_{W2}^{A}(s[X]) \\ &= s\dim_{W2}^{A}(X) \end{aligned}$$

Thus $\dim_{W2}^{A}(X) = 1$, as desired. □

Theorem 1.25. *If A fixes a computable weak 2 dimension s, then it also fixes dimension r for all computable $r > s$.*

Proof. Let X have dimension r and a computable s be as in the statement of $(2')$ with

$r > s$. Applying Lemma 1.23 twice and (2′) once, we can see that:

$$\begin{aligned}\frac{s}{r}\dim_{W2}(X) &= \dim_{W2}\left(\frac{s}{r}[X]\right)\\ &= \dim^A_{W2}\left(\frac{s}{r}[X]\right)\\ &= \frac{s}{r}\dim^A_{W2}(X)\end{aligned}$$

Giving $\dim^A_{W2}(X) = r$, as desired. □

We now tackle the relationship between these two lowness properties. Results of Downey, Nies, Weber and Yu [10] and Kjos-Hansen, Miller and Solomon [17] combine to tell us that the lowness properties of the parent randomness notions coincide exactly. With the dimension versions, the lowness properties coincide below $0'$, but not necessarily outside that lower cone.

Here we will need a stronger version of the following alternate characterization of low for dimension from the main theorem of [20]:

Theorem 1.26. *A is low for dimension if and only if it has the Σ^0_1 covering property, i.e., if $W \subseteq 2^{<\omega}$ is A-c.e., $0 \leq s < t \leq 1$, and $DW_s(W) < 1$, then there is a c.e. set V with $DW_t(V) < 1$ such that $[W] \subseteq [V]$.*

See proof in [20].

The stronger version of Theorem 1.26 that we want is the contrapositive of this theorem, replacing direct weight with optimal weight.

Lemma 1.27. *If A is not low for effective dimension, then there is a $W \subseteq 2^{<\omega}$ which is A-c.e. and $0 \leq s < t \leq 1$ such that $DW_s(W) < 1$ and for every c.e. $V \subseteq 2^{<\omega}$ with*

$OW_t(V) < 1$ it follows that $[W] \setminus [V] \neq \emptyset$.

Proof. Let A not low for effective dimension. Then by the converse of Theorem 1.26, there are an A-c.e. collection $W \subseteq 2^{<\omega}$, and $0 < s < r \leq 1$ with $\mathrm{DW}_s(W) < 1$ such that for all c.e. $V \subseteq 2^{<\omega}$ it follows that if $\mathrm{DW}_r(V) < 1$, then $[W] \setminus [V] \neq \emptyset$. Now, we claim that if we have $s < t < r$ then s, t, and W have the desired property.

Let $V \subseteq 2^{<\omega}$ be a c.e. collection such that $\mathrm{OW}_t(V) < 1$. Then there is some $S \subseteq 2^\omega$ with $[V] \subseteq [S]$ with $\mathrm{DW}_t(S) < 1$. Let $\varepsilon = 1 - \mathrm{DW}_t(S)$. Then there is a finite subcollection $S^* \subset S$ with $\mathrm{DW}_t(S \setminus S^*) < \frac{\varepsilon}{C(t,r)}$ where $C(t,r)$ is as in Lemma 1.5. Then we let $V^* = \{\sigma \in V : \neg\exists\tau \in S^* \text{ with } \tau \prec \sigma\}$. We note that V^* is c.e. and has

$$\mathrm{OW}_t(V^*) \leq \mathrm{DW}_t(S \setminus S^*) < \frac{\varepsilon}{C(t,r)}.$$

But then by Lemma 1.5 there is a c.e. U covering V^* with $\mathrm{DW}_r(U) < \varepsilon$, and we have $X = S^* \cup U$ a c.e. collection with $[V] \subseteq [X]$ and $\mathrm{DW}_r(X) < 1$ and so by Lemma 1.26 it follows that $[W] \setminus [X] \neq \emptyset$. □

In comparing the two lowness notions, we start with the containment direction which always holds. We will actually be able to prove something stronger than just the containment in this direction. Defining the dual notion for these two dimension notions as $A \in$ Low(weak 2 dimension, effective dimension) meaning

$$(\forall X \subseteq 2^\omega)(\dim^A(X) \geq \dim_{W2}(X)),$$

the following theorem holds:

Theorem 1.28. *If A is Low(weak 2 dimension, effective dimension), then A is low for* *effective dimension.*

Proof. We prove by contrapositive. Assume that A is not low for effective dimension, then let $W \subseteq 2^{<\omega}$ and $0 < s < t \leq 1$ be as in Lemma 1.27.

We define W^n to be the collection of strings that can be achieved by concatenating exactly n-many strings where each substring is a member of W. That is, $W^1 = W$, and $W^n = \{\sigma \frown \tau \text{ for } \sigma \in W^{n-1}, \tau \in W\}$. Then we will construct an $x \in \bigcap_n [W^n]$ such that $x \notin B_n$ for all $B_n = \bigcap_i B_{i,n}$ where $\{B_{i,n}\}_{i\in\omega}$ is a weak 2 t-test.

Stage 0: Let $X_0 = \emptyset$, $\sigma_0 = \lambda$.

Stage $n+1$: Given a $\sigma_n \in W^n$ and a c.e. X_n with $[X_n] \subset [\sigma_n]$ such that $\mathrm{OW}_t(X_n|\sigma_n) < 1$, and X_n is t-closed, we will avoid the nth weak 2 t-test, $\{B_{i,n}\}_{i\in\omega}$. First, we observe that there must be some $i \in \omega$ such that $\mathrm{DW}_t(B_{i,n}|\sigma_n) < 1 - \mathrm{OW}_t(X_n|\sigma_n)$, otherwise $\mathrm{DW}_t(B_{i,n}) \geq 2^{-|\sigma_n|}(1 - \mathrm{OW}_t(X_n)) > 0$ for all i, but this is impossible as we know this quantity tends to zero.

Then we note that the collections $S_n = \{\tau \in 2^{<\omega} : \sigma_n \frown \tau \in B_{i,n}\}$ and $Y_n = \{\tau \in 2^{<\omega} : \sigma_n \frown \tau \in X_n$ are c.e. and $\mathrm{DW}_t(S_n) + \mathrm{OW}_t(Y_n) < 1$. Then, we can apply Lemma 1.8 from above to find $V_n \subseteq 2^{<\omega}$ a c.e. t-closed cover of $S_n \cup Y_n$ with $\mathrm{OW}_t(V_n) < 1$. Then by Lemma 1.27 $[W] \setminus [V_n] \neq \emptyset$. Let $\sigma_{n+1} \in W^n$ with $\sigma_{n+1} \succ \sigma_n$ such that $[\sigma_{n+1}] \nsubseteq [\sigma_n \frown V_n]$ and let $X_{n+1} = \{\tau \in \sigma_n \frown V_n : \tau \succ \sigma_{n+1}\}$. $\square$

Corollary 1.29. *If A is low for weak 2 dimension, then A is low for effective dimension.*

In order to prove containment the other direction where it holds, we will need the following strengthening of the c.e. covering property from [20].

Lemma 1.30. *Let A low for effective dimension, I an A-c.e. set with members of the* *form* $\langle\sigma,\tau\rangle$ *such that*

$$\sum_{\langle\sigma,\tau\rangle\in I} 2^{-s|\tau|} < 1.$$

Then for all $t > s$*, there exists a c.e.* $J \supseteq I$ *with*

$$\sum_{\langle\sigma,\tau\rangle\in J} 2^{-t|\tau|} < \infty.$$

Proof. Let $t > s$. Given I as above, we can build a request set relative to A by $R^A = \{\langle\langle\sigma,\tau\rangle, s|\tau|\rangle : \langle\sigma,\tau\rangle \in I\}$. By assumption, this set has total weight less than 1, and so by the Kraft-Chaitin Theorem, there is a prefix-free machine M with oracle A with $|M^A\langle\sigma,\tau\rangle| = s|\tau|$ for all $\langle\sigma,\tau\rangle \in I$. Thus, we have for all $\langle\sigma,\tau\rangle \in I$ that $K^A\langle\sigma,\tau\rangle \leq^+ s|\tau|$. Further, since A is low for dimension, it follows by a result of [20] that A is lowish for K, and so for all $\langle\sigma,\tau\rangle \in I$, we have that $K\langle\sigma,\tau\rangle \leq^+ t|\tau|$. Finally, we define $J = \{\langle\sigma,\tau\rangle : K\langle\sigma,\tau\rangle < t|\tau| + c\}$ for c large enough that $I \subseteq J$. This is the desired J, as the Kraft inequality tells us that

$$\sum_{\langle\sigma,\tau\rangle\in J} 2^{-t|\tau|} < 2^c < \infty. \qquad \square$$

We use this property to show that for A which are Δ^0_2, the other direction of containment holds as well.

Theorem 1.31. *If* $A \leq_T 0'$ *and* A *is low for effective dimension, then for all* $t > s$*, every weak 2* s*-null* Π^A_2 *class has a weak 2* t*-null* Π^0_2 *superclass.*

Proof. Let X be a weak 2 s-null Π^A_2 class and $s < r < t$. So $X = \bigcap_{i\in\omega}[X^A_i]$ for Σ^A_1 collections of strings $\{X^A_i\}_{i\in\omega}$ with $\mathrm{DW}_s(X^A_i) = a_i$ such that $\lim_{i\to\infty} a_i = 0$. (Note: we can assume without loss of generality that the $[X_i^A]$ are nested.) Now let $I_i = \{\langle \sigma, \tau \rangle : \tau \in X_i^A \text{ with use } \sigma\}$. Then we apply Lemma 1.30 to the I_i in order to get c.e. $J_i \supseteq I_i$ with the property

$$\sum_{\langle \sigma,\tau\rangle \in J_i} 2^{-r|\tau|} < \infty.$$

By assumption $A \leq_T 0'$, so let $\{A_n\}_{n\in\omega}$ be a computable sequence approximating A. Then we define

$$T_{i,n} = \{\langle \sigma, \tau \rangle \in J_i : (\exists m \geq n)\tau \in X_{i,m}^{A_m} \text{ with use } \sigma\}$$

and let $U_{i,n} = \{\tau : (\exists \sigma)\langle \sigma, \tau \rangle \in T_{i,n}\}$ be the projection of $T_{i,n}$ onto the second coordinate. $\{T_{i,n}\}_{n\in\omega}$ and $\{U_{i,n}\}_{n\in\omega}$ are computable sequences of c.e. sets. Define

$$Y_j = \bigcap_{i+n=j} [U_{i,n}].$$

Then, we claim that $Y = \bigcap_j Y_j$ is the desired Π_2^0 class.

We first note that $X \subseteq Y$. This follows from the fact that the $[X_i^A]$ are nested, and the fact that $X_i^A \subseteq U_{i,n}$ for all i, n. We can see that the latter statement must be true because of the fact that for any $\tau \in X_i^A$, there is a use $\sigma \prec A$ witnessing $\tau \in X_i^\sigma$, and so $\langle \sigma, \tau \rangle \in I_i \subseteq J_i$, and also since A_n is a computable approximation of A, it follows that it eventually gets σ right and stops changing, so for all n, there will always be $m > n$ putting $\langle \tau, \sigma \rangle \in T_{i,n}$, and so $\tau \in U_{i,n}$, as desired.

Finally, we must show that Y is weak 2 t-null. It suffices to demonstrate that $\lim_{j\to\infty} \mathrm{OW}_r(Y_J) = 0$. Given this, we can apply Lemma 1.6 to each Y_i to get a sequence of c.e. $V_i \subseteq 2^{<\omega}$ with $Y \subseteq Y_i \subseteq [V_i]$ for all i. Further, since $\mathrm{DW}_t(V_i) \leq C(r,t) \cdot \mathrm{OW}_r(Y_i)$

and the latter tends to zero, we also have that $\lim_{i\to\infty} \mathrm{DW}_t(V_i) = 0$.

We note that $\mathrm{OW}_r(Y_j) \leq \inf\{\mathrm{OW}_r(U_{i,n}) : i+n = j\}$ and $\lim_{i\to\infty} \mathrm{DW}_r(X_i) = 0$. Thus, to prove that the optimal r-weight of Y_j tends to zero, it suffices to show that $\mathrm{DW}_r(U_{i,n}) \to \mathrm{DW}_r(X_i)$.

To prove this, we observe that for every i and every $\langle\sigma,\tau\rangle \in J_i \setminus I_i$, there is an m such that for all $n > m$, $\langle\sigma,\tau\rangle \notin T_{i,n}$. This is true as A_i is a computable approximation of A which means that for any length of initial segment, it is eventually right and stops changing. Thus, if $\sigma \not\prec A$, $\langle\sigma,\tau\rangle$ will eventually stop showing up in $T_{i,n}$. Alternatively, if $\sigma \prec A$ is a true initial segment, then if $\langle\sigma,\tau\rangle \notin I_i$, that means that the enumeration of X_i with use σ does not see τ entering, and so also $\langle\sigma,\tau\rangle \notin J_i$.

Now, let $\varepsilon > 0$, then since

$$\sum_{\langle\sigma,\tau\rangle\in J_i} 2^{-r2|\tau|} < \infty,$$

There is a finite collection $A_{i,\varepsilon} \subseteq J_i \setminus I_i$ such that

$$\sum_{\langle\sigma,\tau\rangle\in J_i\setminus(I_i\cup A_{i,\varepsilon})} 2^{-r2|\tau|} < \varepsilon.$$

Then, since the members of $A_{i,\varepsilon}$ are not in I_i, for each one there is an m so that for $n > m$ they are not contained in $T_{i,n}$. We need only take M the maximum such m and then for all $n > M$ we have $\mathrm{DW}_r(U_{i,n}) \leq \mathrm{DW}_r(X_i) + \varepsilon$. Thus it follows that

$$\lim_{n\to\infty} \mathrm{DW}_r(U_{i,n}) = \mathrm{DW}_r(X_i),$$

as desired. □

Corollary 1.32. *If A is low for effective dimension and $A \leq_T 0'$, then A is low for* *weak 2 dimension.*

Proof. This follows immediately from Theorem 1.31. In particular, if a real X is weak 2 s-null relative to A, then it is weak 2 t-null for all $t > s$, thus we have $\dim_{W2}(X) \leq \dim_{W2}^{A}(X)$. □

It turns out, however, that outside of Δ_2^0, this direction of containment does not necessarily hold. We will use the Y which is described in the following lemma to construct such an oracle.

Lemma 1.33. *There is a Y which is Δ_2^0, low for effective dimension, but not K-trivial.*

Proof. To prove that such a thing exists, we apply a result which appears in both [11] and [20] that there is a Π_1^0 perfect class, all of whose elements are low for dimension. To find our Y we can use $0'$ to construct the tree and then simply build Y by searching for incremental extensions. We start with the $\sigma_0 = \lambda$, then at the n^{th} step, we search for an extension $\sigma_n \succeq \sigma_{n-1}$ with $K(\sigma_n) \geq K(|\sigma_n|) + n$. There must always exist such an extension, as the tree contains uncountably many branches above each node, the K-trivials are countable, and any non-K-trivial satisfies the desired property for some finite initial segment. Then $Y = \bigcup_n \sigma_n$. □

Definition 1.34 (Nies)**.** A set $A \in 2^\omega$ is *LR-reducible* to a set B (denoted by $A \leq_{LR} B$) if every set which is Martin-Löf random relative to B is ML random relative to A.

Similarly, a set $A \in 2^\omega$ is *LK-reducible* to a set B (denoted by $A \leq_{LK} B$) if for all $\sigma \in 2^{<\omega}$, $K^A(\sigma) \geq^* K^B(\sigma)$.

Lemma 1.35 (Nies)**.** *The reducibilities $\leq_{LK}$ and $\leq_{LR}$ are equivalent.*

We now present a theorem of Barmpalias, Miller, and Nies. This is Theorem 4.7 in [2].

Theorem 1.36. *If Y is Δ^0_2 and not K-trivial, then for all $Z \geq_T 0'$, there exists $X \leq_{LR} Y$ such that $X \oplus 0' \equiv_T Z$.*

We will apply this theorem with the specific Y whose existence we proved above to show that the two lowness notions do not always coincide. This is a more specific version of Corollary 4.8 from [2].

Corollary 1.37. *There exists $X (\not\leq_T 0')$ with X low for effective dimension, but not low for weak 2 dimension.*

Proof. Let Y be as in Lemma 1.33, then we we apply Theorem 1.36 with $Z = 0''$. This gives us X with $X \leq_{LR} Y$ and so $X \leq_{LK} Y$, but since Y is low for dimension, so is X and $X \oplus 0' \equiv_T Z$. Now let $A < 0''$ be 2-random. We note that this A is also a weak 2-random and so weak 2 dimension 1. However,

$$A \leq_T 0'' \equiv_T X \oplus 0' \leq_T X'.$$

We have that A is Δ^X_2, and hence, by the relativization of Theorem 1.10, has weak 2 dimension 0 relative to X. Thus X fails (badly) to be low for weak 2 dimension. □

We note that as a consequence of these results, the other characterizations of low for dimension from [20] ((4)-(7) of the main theorem) also characterize low for weak 2 dimension for $A \leq_T 0'$. However, outside of this set, they do not necessarily coincide, although one direction of the implications is always true.

Chapter 2

Computable analogs of lesser-known cardinal characteristics

All work in this Chapter was done jointly with Ivan Ongay-Valverde.

2.1 Introduction

Recent work of Rupprecht [30] and, with some influence of Rupprecht but largely independently, Brendle, Brooke-Taylor, Ng, and Nies [6] developed and showed a process for extracting the combinatorial properties of cardinal characteristics and translating them into highness properties of oracles with related combinatorial properties. Some of the analogs so derived are familiar computability-theoretic properties, some are new characterizations of existing notions, and some are completely new. The remarkable part of this work is that many of the proofs of relationships between the cardinals in the set-theoretic setting translate almost perfectly to the effective setting. The work so far has mostly focused on the cardinal characteristics of Cichoń's diagram.

The nodes in Cichoń's diagram are defined in a couple of related collections, with the unbounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$ defined as follows:

Definition 2.1. For $f, g \in \omega^\omega$, we say f *dominates* g if $f \geq g$ almost everywhere, that

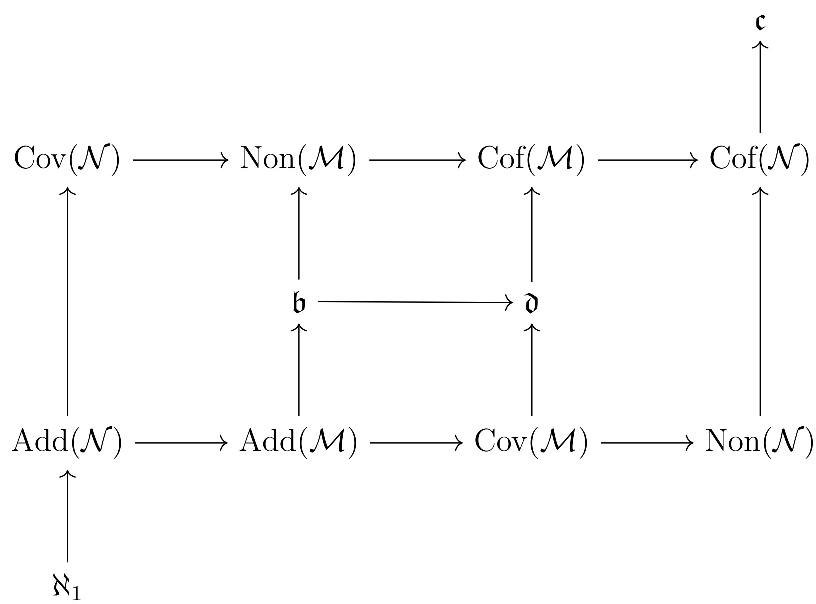

Figure 1: Cichoń's diagram

is $f \geq^* g \iff \forall^\infty n \in \omega\ f(n) \geq g(n)$.

Then $\mathfrak{b}$, the *unbounding number*, is the smallest number of functions not dominated by any one function, that is $\min\{|\mathcal{A}| : \mathcal{A} \subset \omega^\omega$ such that $\forall f \in \omega^\omega\ \exists g \in \mathcal{A}\ f \not\geq^* g\}$.

Similarly, $\mathfrak{d}$, the *dominating number*, is the smallest number of functions guaranteed to dominate any other function, that is $\min\{|\mathcal{A}| : \mathcal{A} \subset \omega^\omega$ such that $\forall f \in \omega^\omega\ \exists g \in \mathcal{A}\ f \leq^* g\}$.

Then, the remaining nodes are all characteristics of ideals of the real line.

Definition 2.2. If $\mathcal{I} \subset \mathbb{R}$ is an ideal, we define:

$\mathrm{Add}(\mathcal{I})$ is the smallest collection of members of the ideal that union to something not in the ideal, that is

$$\mathrm{Add}(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}, \bigcup_{A \in \mathcal{A}} A \notin \mathcal{I}\right\}.$$

Cof($\mathcal{I}$) is the smallest collection of members of the ideal that contains a member covering any member of the ideal, that is

$$\mathrm{Cof}(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}, \forall X \in \mathcal{I}\ \exists A \in \mathcal{A}\ X \subset A\right\}.$$

Non($\mathcal{I}$) is the size of the smallest set not contained in the ideal, that is

$$\mathrm{Non}(\mathcal{I}) = \min\left\{|A| : A \subset \mathbb{R}, A \notin \mathcal{I}\right\}$$

Cov($\mathcal{I}$) is the size of the smallest collection of members of the ideal that covers the whole space, that is

$$\mathrm{Cov}(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}, \bigcup_{A \in \mathcal{A}} A = \mathbb{R}\right\}$$

The remaining 8 nodes are one of these forms for the ideals $\mathcal{N}$, the Lebesgue null sets, or $\mathcal{M}$, the meager sets.

The arrows here stand for inequalities, with $A \to B$ in the diagram indicating $A \leq B$.

There is a purely semantic formulation of the translation scheme to an effective notion where all of these characteristics can be viewed as either an unbounding number or a dominating number along the lines of $\mathfrak{b}$ and $\mathfrak{d}$ for a different relationship between two spaces. They can then be semantically converted to the appropriate highness notion. For all the details of the semantic scheme, see [30] or [6].

An alternative, somewhat intuitive way to think about this translation scheme is to frame it as follows: When working with cardinal characteristics on the set theory side, it's common to build models by forcing extensions that have specific properties. One common trick is to force a characteristic to be larger by building an extension which has a new object that negates the desired property for a specific collection from the ground model. If we reinterpret the ground model as the computable objects, and the extension as adding those things computable from an oracle, the highness property corresponding to the characteristic will be exactly the combinatorial property needed to negate the characteristic property for the collection of computable objects. Among other things, this means that the highness notions actually end up looking like the negations of the characteristics that they were derived from.

For example, let's take the unbounding number $\mathfrak{b}$. In building a forcing extension to make $\mathfrak{b}$ larger, we would want to add a function which *does* bound a collection of functions from the ground model. When translated to a computability-theoretic highness notion, this becomes an oracle which computes a function dominating every computable function. This is exactly the set of oracles of high degree. Similarly, for the domination number $\mathfrak{d}$, in building a forcing extension to make $\mathfrak{d}$ larger, we would want to add a function which is not dominated by any of a collection of functions from the ground model. When translated to the computability side, this becomes an oracle which computes a function not dominated by any computable function, i.e. of hyperimmune degree. Some of the analogs, like these, are well-studied, and some were introduced by Rupprecht in [30].

Translating the remaining characteristics will require introducing effective versions of null sets and meager sets, as well as some related notions. We present those here:

Definition 2.3. A *Schnorr test* is a uniformly Σ^0_1 sequence of sets $\{A_n\}$ such that $\mu(A_n) = \frac{1}{2^n}$.

$X \in 2^\omega$ is *Schnorr random* iff $X \notin \bigcap A_n$ for all Schnorr tests $\{A_n\}$.

$X \in 2^\omega$ if *Schnorr engulfing* if there is a Schnorr test relative to X covering all Schnorr tests relative to 0.

$X \in 2^\omega$ is *weakly Schnorr engulfing* iff it computes a Schnorr test containing all computable reals.

$X \in 2^\omega$ is *low for Schnorr tests* iff every Schnorr test relative to X is covered b an unrelativized Schnorr test.

Definition 2.4. An *effectively meager set* is a uniform union of nowhere dense Π^0_1 classes.

$X \in 2^\omega$ is *weakly 1-generic* iff X escapes all effectively meager sets.

$X \in 2^\omega$ is *meager engulfing* if there is an X-effectively meager set containing all effectively meager sets.

$X \in 2^\omega$ is *weakly meager engulfing* iff it computes an effectively meager set containing all computable reals.

$X \in 2^\omega$ is *low for weak 1-generic* iff the weak 1-generics relative to X are the weak 1-generics.

We will later use the following equivalences:

Theorem 2.5 (Rupprecht [30]). *The following are equivalent for $A \in 2^\omega$:*

(1) A is Schnorr engulfing;

(2) A meager engulfing;

(3) A computes a high degree.

Theorem 2.6 (Kurtz [19]). *The following are equivalent for $A \in 2^\omega$:*

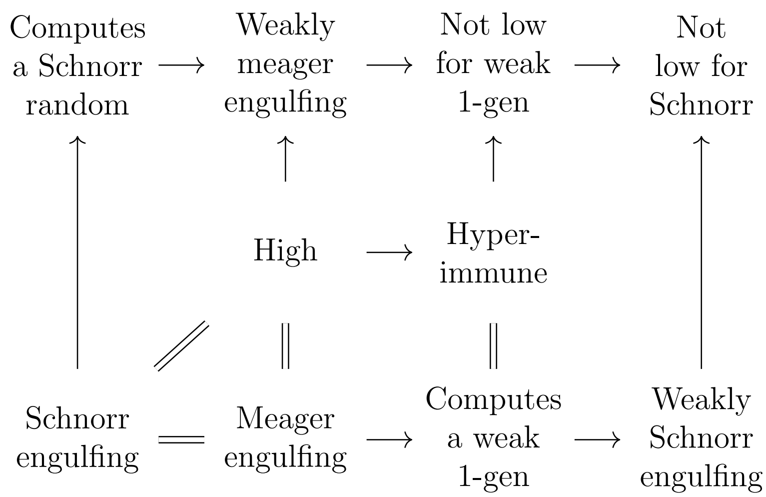

Figure 2: Effective Cichoń's diagram

(1) A computes a weak 1-generic.

(2) A is of hyperimmune degree.

In the effective diagram arrows actually do mean implication, where the lower-left highness properties are generally stronger than the upper-right.

In this thesis, we will expand on this work by looking at four of different cardinal characteristics not appearing in Cichoń's diagram. First, we will examine the evasion number, a cardinal characteristic first introduced by Blass in [3], as well as its less-studied dual, the prediction number. We will also look at two forms of the so-called rearrangement number, as introduced by Blass et al. in [4]. In all these cases, we will give the correct effective analogs, and prove relationships between these new highness notions and their relationships with other highness notions which are analogous to well-studied cardinal characteristics.

2.2 Prediction and Evasion

2.2.1 Definitions

Definition 2.7 (Blass [3]). A *predictor* is a pair $P = (D, \pi = \langle \pi_n : n \in D \rangle)$ where $D \in [\omega]^\omega$ (infinite subsets of ω) and where each $\pi_n : \omega^n \to \omega$. By convention, we will sometimes refer to $\pi_n(\sigma)$ by simply $\pi(\sigma)$. This predictor P *predicts* a function $x \in \omega^\omega$ if, for all but finitely many $n \in D, \pi_n(x\restriction_n) = x(n)$. Otherwise x *evades* P. The *evasion number* $\mathfrak{e}$ is the smallest cardinality of any family $E \subseteq \omega^\omega$ such that no single predictor predicts all members of E.

We will also make use of the dual to $\mathfrak{e}$, which is explored by Brendle and Shelah in [7].

Definition 2.8. The *prediction number*, which we will call $\mathfrak{o}$, is the smallest cardinality of any family O of predictors such that every function is predicted by a member of O.

The known results for $\mathfrak{e}$ and $\mathfrak{o}$ position them as illustrated in figure 3 relative to Cichoń's diagram.

In order to effectivize our prediction-related cardinal characteristics, we must first effectivize the definition of a predictor.

Definition 2.9. A *computable predictor* is a pair $P = (D, \langle \pi_n : n \in D \rangle)$ where $D \subseteq \omega$ is infinite and computable and each $\pi_n : \omega^n \to \omega$ is a computable function.

Similarly, we define an *A-computable predictor* as the relativized version where all objects are computable relative to some oracle A.

Finally, we define an oracle A to be of *evasion degree* if there is a function $f \leq_T A$ which evades all computable oracles, and A is of *prediction degree* if there is a predictor

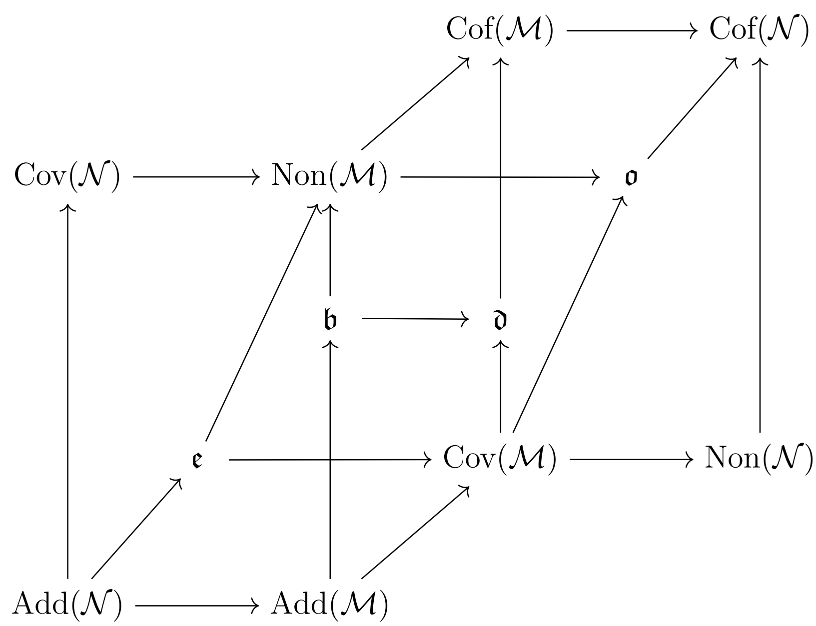

Figure 3: Cichoń's diagram including $\mathfrak{e}$ and $\mathfrak{o}$.

$P \leq_T A$ which predicts all computable functions.

Because of the fact that we negate the original statements of the definitions of cardinal characteristics, under our scheme the evasion number $\mathfrak{e}$ is an analog to being a prediction degree, and the prediction number $\mathfrak{o}$ is an analog to being an evasion degree.

We present below known facts about $\mathfrak{e}$ and $\mathfrak{o}$ represented by Cichoń's diagram with $\mathfrak{e}$ and $\mathfrak{o}$ included, as well as their translations into effective analogs.

Theorem 2.10. *The following relationships are known for* $\mathfrak{e}$*.*

Cardinal Char.	*Highness Properties*	*Theorem*
add($\mathcal{N}$) $\leq \mathfrak{e}$ *[3]*	*Schnorr engulfing* $\Rightarrow$ *prediction degree*	*2.11*
$\mathfrak{e} \leq$ *non*($\mathcal{M}$) *[15]*	*prediction degree* $\Rightarrow$ *weakly meager engulfing*	*2.13*
$\mathfrak{e} \leq$ *cov*($\mathcal{M}$) *[3]*	*prediction degree* $\Rightarrow$ *weakly 1-generic*	*2.14*
CON($\mathfrak{e} <$ *add*($\mathcal{M}$)) *[5]*	*meager engulfing* $\not\Rightarrow$ *prediction degree*	*False*
CON($\mathfrak{b} < \mathfrak{e}$) *[7]*	*prediction degree* $\not\Rightarrow$ *high*	*2.15*

Similarly, for $\mathfrak{o}$ *(all results can be found in [7])*

Cardinal Char.	*Highness Properties*	*Theorem*
cov($\mathcal{M}$) $\leq \mathfrak{o}$	*weakly 1-generic* $\Rightarrow$ *evasion degree*	*2.17*
non($\mathcal{M}$) $\leq \mathfrak{o}$	*weakly meager engulfing* $\Rightarrow$ *evasion degree*	*2.19*
$\mathfrak{o} \leq$ *cof*($\mathcal{N}$)	*evasion degree* $\Rightarrow$ *not low for Schnorr tests*	*2.22*
CON(*cof*($\mathcal{M}$) $< \mathfrak{o}$)	*evasion degree* $\not\Rightarrow$ *not low for 1-generics*	*Open*
CON($\mathfrak{o} < \mathfrak{d}$)	*hyperimmune* $\not\Rightarrow$ *evasion degree*	*False*
–	*not low for Schnorr Tests* $\not\Rightarrow$ *evasion*	*2.28*

2.2.2 Prediction Degrees

Theorem 2.11. *If $A \in 2^\omega$ is high, then it is of prediction degree.*

Proof. Let A be high and set $D = \omega$. We will use the fact that if A is high, then A can enumerate a list of indices for the total computable functions. A proof of this fact can be found in [13]. Using this, we simply enumerate all the computable functions. Then to define π_n, for each finite string $f \in \omega^n$, we go through the list of computable functions $\{\varphi_e\}$ until we find one such that $\varphi_e \upharpoonright n = f$. Then we define $\pi_n(f) = \varphi_e(n)$. This predictor is computable in A and predicts all computable functions. □

Lemma 2.12. *For any predictor P, there is an effectively-in-P meager set covering all functions predicted by P.*

Proof. The collection

$$C_i = \{f : |\{n \in D : \pi(f\restriction_n) \neq f(n)\}| < i\}$$

is nowhere dense and Π^0_1 in P, and the collection of functions predicted by P is exactly $\bigcup_{i \in \mathbb{N}} C_i$. □

Theorem 2.13. *If A is a prediction degree, then A is weakly meager engulfing.*

Proof. Assume A is a prediction degree, then there is a predictor $P \leq_T A$ which predicts all computable functions. In particular, we just need a predictor which predicts all $0,1$-valued computable functions.

Then, by Lemma 2.12 one can, using P, effectively find a meager set covering every function predicted by P. Thus there is an A-effectively meager set covering all $0,1$-valued computable functions, and hence covering all computable reals, as desired. □

Theorem 2.14. *If $A \in 2^\omega$ is of prediction degree, then A is weakly 1-generic.*

We will actually prove the equivalent statement that if A is a prediction degree, then A has hyperimmune degree. This is an analog of the characteristic inequality $\mathfrak{e} \leq \mathfrak{d}$. The above theorem is the analog of the strictly stronger cardinal relation $\mathfrak{e} \leq \text{cov}(\mathcal{M})$. However, these notions are one of the places where a relationship that is separable in the set-theoretic case collapses in the computability-theoretic analog, so the theorems are equivalent. The proof follows one of Blass from [3].

Proof. Given $A \in 2^\omega$ which is not weakly 1-generic, by a result of Kurtz, A is hyperimmune-free. In particular we will use the fact that for all $f \leq_T A$ with $f : \omega \times \omega \to \omega$, there is a function $g \leq_T 0$ such that $g > f$.

Let $P = (D_P, \{\pi_n\}) \leq_T A$ be a predictor, and define $f : \omega \times \omega \to \omega$ by

$$f(n,k) = \begin{cases} \max\{\pi_n(t) : t \in \omega^n \text{ and } t_i < k \text{ for all } i \in n\} & \text{if } n \in D_P \\ 0 & \text{otherwise} \end{cases}$$

We note that $f \leq_T A$. Then, by assumption, there is a computable function g such that $g(n,k) > f(n,k)$ for all n, k. Then we define

$$x(n) = g(n, 1 + \max\{x(p) : p < n\}).$$

Now, let $n \in D_\pi$ and $k = 1 + \max\{x(p) : p < n\}$. We note that $x\restriction_n$ is of length n and has all values less than k, and so is an admissible t from the definition of $f(n,k)$, so $f(n,k) \geq \pi_n(x\restriction_n)$. On the other hand, by definition of x and the choice of g, we also have $x(n) \geq g(n,k) > f(n,k)$. Thus, we have $x(n) > \pi_n(x\restriction_n)$. Since n was arbitrary, it follows that x evades P, and so A is not a prediction degree. □

Theorem 2.15. *There is an A which is of prediction degree but does not compute any B which is high.*

Proof. We will force with conditions $\langle d, \pi, F\rangle = p \in P$ where $d \in 2^{<\omega}$ is a finite partial function, $\pi = \{\pi^n : n \in d\}$ and $\pi^n : \omega^n \to \omega$ is a finite partial function, $F \subset \omega^\omega$ is a finite collection of functions with the property $f, g \in F, f \neq g \Rightarrow f\restriction_{|d|} \neq g\restriction_{|d|}$. Here, the

d and π can be thought of as partial approximations of D and π in the eventual predictor we are constructing, and F as the collection of functions that we are committed to predicting correctly for the rest of the construction.

We define (d', π', F') as an extension of (d, π, F) by

$$(d', \pi', F') \leq (d, \pi, F) \iff d' \supset d, \pi' \supset \pi, F' \supset F \text{ and}$$
$$f \in F, n \in \mathrm{dom}(d') \setminus \mathrm{dom}(d) \Rightarrow \pi'^n(f\restriction_n) = f(n)$$

During this construction, we will also maintain the property that $\bigoplus F_s = \bigoplus_{f \in F_s} f$ is hyperimmune-free.

To initialize the construction, we let $d_0 = \langle\rangle, \pi_0 = \{\}, F_0 = \{\}$.

We will extend in our construction by the following rules:

P_e: The goal of this requirement will be to ensure that we predict φ_e.

At stage $s = 3e$, we simply set $F_s = F_{s-1} \cup \{\varphi_e\}$ and $d_s = d_{s-1} {}^\frown 0^n$ with n least such that for $f \in F_{s-1}$, if $\varphi_e \neq f$, then $\varphi_e\restriction_{|d_s|} \neq f\restriction_{|d_s|}$. Additionally, if $\pi_s^n \in \pi_s$ has that $\pi_s^n(\varphi_e\restriction_n)$ is undefined, we define it to be $\varphi_e(n)$.

I_e: The goal of this requirement is to ensure that D is infinite.

At stage $s = 3e+1$, $D_s = D_{s-1} {}^\frown 1$, and $\pi_s = \pi_{s-1} \cup \{\pi^m\}$ with $m = |D_{s-1}|$ where $\pi^m : \omega^m \to \omega$ with $\pi^m(f\restriction m) = f(m)$ for all $f \in F_{s-1}$, $\pi^m(\sigma) = 0$ for all other σ, and $F_s = F_{s-1}$.

$E_{e,n}$: The goal of this requirement will be to ensure that φ_e^A is not total or that there is a computable function h_e such that $\exists^\infty n(\varphi_e^A(n) \leq h_e(n))$.

At stage $s = 3\langle e, 0\rangle + 2$, we will use the following claim:

Claim 2.16. *We claim that either*

(1) There is a uniformly $\Sigma_1^{0,\bigoplus F_s}$ *collection of functions indexed by* $d, \pi, \overline{f}^*$ *where* d, π *are as in* P *and* $\overline{f}^*$ *is finite initial segments of functions. These functions have the property that for any collection* $\hat{F}$ *of total functions extending the* $\overline{f}^*$*, the forcing condition* $\langle d, \pi, F_s \cup \hat{F}\rangle$ *can be extended by* $q = \langle d_q, \pi_q\rangle(n)$ *is below our function. Syntactically, this is*

$$\{h^e_{d,\pi,\overline{f}^*} \in \omega^\omega : \langle d, \pi, F_s\rangle \leq \langle d_s, \pi_s, F_s\rangle, \overline{f}^* = \langle f_i^*\rangle, |\overline{f}^*| = l \in \omega, f_i^* \in \omega^{|d|}$$
$$\textit{are distinct and } \forall f \in F_s, f_i^* \in \overline{f}^*\} f_i^* \neq f\restriction_{|d|}$$

such that

$$h^e_{d,\pi,\overline{f}^*}(n) \geq \min\{m : \forall p = \langle d, \pi, F_s \cup \hat{F}\rangle \textit{ with } \hat{F} = \{f_i \in \omega^\omega\}_{i<l} \textit{ and } f_i\restriction_{|d|} = f_i^*$$
$$\exists q \leq p \quad \varphi_e^{\langle d_q, \pi_q\rangle}(n) \downarrow < m\}$$

or,

(2) There is $n \in \omega$ *and* $p \leq \langle d_s, \pi_s, F_s\rangle$ *such that for any* $q \leq p$*,* $\varphi_e^{\langle d_q, \pi_q\rangle}(n) \uparrow$ *and* $\bigoplus F_p$ *is hyperimmune free.*

If (2), then we define $\langle d_{s+1}, \pi_{s+1}, F_{s+1}\rangle$ to be such a p and we do nothing for stages of the form $s = 3\langle e, n\rangle + 2$. This will make $\varphi_e^{\langle d,\pi\rangle}$ not total.

If (1), then we can find $\widehat{h}^e \leq_T \bigoplus F_s$ such that $\widehat{h}^e \geq^* h^e_{d,\pi,\overline{f}^*}$ for all such functions. However, since this join is hyperimmune-free, it follows that there is a computable function h^e for which $(\forall n) h^e(n) \geq \widehat{h}^e$.

At stage $s = 3\langle e, n+1\rangle + 2$ we can find $j > n$ so that $h^e(j) \geq h^e_{d_s,\pi_s,\overline{f}^*}(j)$ where $\overline{f}^*$ are the restrictions of the functions in $F_s \setminus F_{3\langle e,0\rangle+2}$ to $|d_s|$ and such that $\varphi_e^{\langle d_s,\pi_s\rangle}(j)$ is not yet defined.

In this situation, we can find $p_{s+1} = \langle d_{s+1}, \pi_{s+1}, F_{s+1}\rangle$ such that $\varphi_e^{\langle d_{s+1},\pi_{s+1}\rangle}(j) \downarrow\leq h^e_{d_s,\pi_s,\overline{f}^*}(j) \leq h^e(j)$, however, we note that this property of the p_{s+1} only depends on finite initial segments of the the members of $F_{s+1} \setminus F_s$, and so there actually is such a condition with $\bigoplus F_{s+1}$ hyperimmune-free. We pick a condition with this property.

Verification: By construction, the predictor $P = \langle \bigcup d_s, \bigcup \pi_s\rangle$ has the desired properties. P_e ensures our predictor predicts all computable functions, I_e ensures that $(\bigcup d_s)^{-1}(1)$ is infinite, and $E_{e,n}$ ensures that the computational strength of the predictor cannot compute a total function dominating the computable functions, and so P is not high. $\square$

Proof of Claim 2.16:

Proof. Before doing the technical work to show the claim, we will explain the idea of the upcoming proof. As we see above, we want – if possible – to define the function h^e_{d,π,g_i^*} in such a way that, given $\langle d, \pi, F_s \cup G\rangle \leq \langle d_s, \pi_s, F_s\rangle$ with $G = \{g_i : i < l+1\}$ and $g_i\restriction_{|d|} = g_i^*$ then we can find $q \leq \langle d, \pi, F_s \cup G\rangle$ such that $\varphi_e^{\langle d^q,\pi^q\rangle}(n)$ is smaller than $h^e_{d,\pi,\langle g_i^*:i<l+1\rangle}(n)$. In other words, $h^e_{d,\pi,g_i^*}(n)$ represents the minimal value that we can force $\varphi_e^{\langle D,\pi\rangle}(n)$ to take given that we already committed to d, π, g_i^*.

In order to do this, we try to find all the possible extensions of $\langle d, \pi, F_s \cup G\rangle$ that make $\varphi_e^{\langle d^q,\pi^q\rangle}$ small. In general this is not necessarily possible, but our best chance to find them is if we restrict ourselves to a compact space (there, we will only have finitely many extensions that are compatible with everything to consider).

The conversion from the whole ω^ω to a compact space is possible thanks to the following observation: $q = \langle d^q, \pi^q, F^q \rangle \leq \langle d, \pi, F \rangle$ and $\langle d, \pi, \{g\} \rangle$ are compatible if $g \upharpoonright |d|$ is different from $f \upharpoonright |d|$ for all $f \in F$ and $g(|d|)$ is bigger than the $|d|$th index of all strings in the domain of any function in π^q (more formally, it is bigger than $\sigma(|d|)$ for all $\sigma \in dom(\pi^{n,q})$ with $d(n) = 1$).[1] This observation hints at the possibility of only worrying about functions of certain growth while we are looking for our small convergences.

In our proof, we will ask $h^e_{d,\pi,g^*_i}(n)$ to not only be bigger than the minimal value that $\varphi_e^{\langle D,\pi \rangle}$ can take, but also to be bigger than the values taken by strings in the domain of functions from π. In that way, we make $h^e_{d,\pi,g^*_i}(n)$ carry some information of compatibility. To define the compact space where we will work, we will define functions B_l that combine nicely the information needed.

Now, for the proof, we will show this by induction on $l = |\overline{f}^*|$. Our induction hypothesis is slightly stronger than the statement of the claim. Case (2) remains unchanged, but we add to case (1) the additional requirement:

(1a) For all $f \in \omega^\omega$ with $f(n) > h^e_{d,\pi,\overline{f}^*}(n)$ for $n < |d|$ then $\langle d, \pi, F_s \cup \{f\} \rangle$ is compatible with an extension $r \leq \langle d, \pi, F_s \cup \widehat{F} \rangle$ with $\varphi^{\langle d^r, \pi^r \rangle}(n) \downarrow < h^e_{d,\pi,\overline{g}^*}(n)$, $\hat{F} = \{f_i \in \omega^\omega\}_{i<l}$, and $f_i \upharpoonright |d| = f^*_i$. Furthermore, r does not depend on f.

Case $l = 0$:

Fix $\langle d, \pi, F_s \rangle \leq \langle d_s, \pi_s, F_s \rangle$.

We will define a function $h^e_{d,\pi,\emptyset}$ computable from $\bigoplus F_s$ with the desired properties.

Fix $n \in \omega$. For each $k \in \omega$ we will look for $q_k \leq \langle d0^k1, \pi, F_s \rangle$ such that $\varphi^{\langle d_{q_k}, \pi_{q_k} \rangle}(n) \downarrow$.

[1] The compatibility is true because $g \upharpoonright k$ with $k > |d|$ is not mentioned in any function from π^q, therefore, we can create π' which always predicts g correctly after $|d|$ such that $\pi^q \subseteq \pi'$. In this way $\langle d^q, \pi', F \cup \{g\} \rangle$ is below q and $\langle d, \pi, \{g\} \rangle$.

If there is k_0 such that the above do not hold, then we have that $p = \langle d0^{k_0}1, \pi, F_s \rangle$ satisfies (2).

Otherwise, we begin searching for extensions $q \leq \langle d, \pi, F_s \rangle$ with $\varphi_e^{\langle d^q, \pi^q \rangle}(n) \downarrow$. As soon as we find a convergence to a value m, we let

$$h^e_{d,\pi,\emptyset}(n) = \max\{m, \min\{k : \forall i \in d^q \ \forall \sigma \in \mathrm{dom}(\pi^q_i) \ \sigma \in k^i\}\}$$

Notice that the first part of the max ensures (1), and the second part ensures that (1a) is satisfied, as the only way there is no such extension, is if π^q incorrectly predicts f for some $n \in (|d|, |d^q|]$, but this is impossible, as f takes a value at some $m < |d|$ which is larger than anything that shows up in the domain of any of the functions from π^q, by definition.

Case $l = 1$:

If (2) already has already happened, we are done. Otherwise, fix $\langle d, \pi, F_s \rangle \leq \langle d_s, \pi_s, F_s \rangle$ and $g^* \in \omega^{|d|}$ such that for all $f \in F_s$, $f\restriction_{|d|} \neq g^*$.

We will define a function $h^e_{d,\pi,\langle g^* \rangle}$ computable from $\bigoplus F_s$ with the desired properties.

Now, let $h^e_{d0^k1,\pi,\emptyset}$ be as in the $l = 0$ case. We define

$$B_1(j) = \begin{cases} 0 & j < |d| \\ \max\{h^e_{d0^k1,\pi,\emptyset}(j) : |d| + k = j\} & |d| \leq j \end{cases}$$

Notice that, given $f \in \omega^\omega$ with $f\restriction|d| = g^*$, if there is $j \geq |d|$ such that $f(j) > B_1(j)$ then $\langle d, \pi, F_s \cup \{f\} \rangle$ is compatible with an extension $r \leq \langle d0^k1, \pi, F_s \cup \emptyset \rangle$ with $\varphi^{\langle d^r, \pi^r \rangle}(j) \downarrow < h^e_{d,\pi,\emptyset}(j)$, for $k = j - |d|$.

Since B_1 is computable from $\bigoplus F_s$ we have that the space

$$C_1 = \{f \in \omega^\omega \ : \ f \upharpoonright |d| = g^* \ \& \ \forall j \geq |d| \ f_i(j) \leq B_1(j)\}$$

is effectively compact with respect to $\bigoplus F_s$.

Fix $n \in \omega$.

Then we can define open sets in C_1 representing bounded convergence. We define these sets as

$$U_m^n = \{h \in C_1 \ : \ \exists q \leq \langle d, \pi, F_s \cup \{h\}\rangle \ \varphi_e^{\langle d_q, \pi_q \rangle}(n) \downarrow < m\}.$$

Notice that $U_m^n \subseteq U_t^n$ as long as $m \leq t$, and that U_m^n is a $\Sigma_1^{0, \bigoplus F_s}$ set of functions.

Furthermore, if we call $A^n = \bigcup_{m \in \omega} U_m^n$, we have that $C \setminus A^n$ is a $\Pi_1^{0, \bigoplus F_s}$ class that can be express as follows:

$$\{h \in C_1 : \forall q \leq \langle d, \pi, F_s \cup \{h\}\rangle \ \varphi_e^{\langle d_q, \pi_q \rangle}(n) \uparrow\}.$$

If $C_1 \setminus A^n \neq \emptyset$, using the hyperimmune-free basis theorem, we can find an h which is hyperimmune-free relative to $\bigoplus F_s$, but since this join is hyperimmune-free, it follows that h is hyperimmune-free, and we can satisfy (2) with $p = \langle d, \pi, F_s \cup \{h\}\rangle$.

Otherwise, $C_1 = A^n = \bigcup_{m \in \omega} U_m^n$, so, by compactness there is m^*, which can be found in an effective way from $\bigoplus F_s$, such that $C_1 = U_{m^*}^n$. This m^* will help us satisfy (1).

Now, in order to satisfy (1a), notice that the set of functions in C^1 where adding them to F_s still allows for an extension witnessing a small convergence for a fixed d', π'

$$O_{d',\pi'} = \{f \in C_1 : \exists \langle d', \pi', F\rangle \leq \langle d, \pi, F_s \cup \{f\}\rangle \; \varphi_e^{\langle d',\pi'\rangle}(n) \downarrow < m^*\}$$

is $\Sigma_1^{0,\bigoplus F_s}$ and that

$$C_1 = U_{m^*}^n = \bigcup_{\langle d',\pi',\emptyset\rangle \in \mathbb{P}} O_{d',\pi'}.$$

By effective compactness we can find $c \in \omega$ and $\langle d^a, \pi^a\rangle$ for all $a \leq c$, such that $C_1 = \bigcup_{a=1}^{c} O_{d^a,\pi^a}$. In other words, this gives us finitely many $\langle d^a, \pi^a, \emptyset\rangle$ forcing a convergence less than m^* compatible with $\langle d, \pi, F_s \cup \{f\}\rangle$ for all $f \in C_1$. Let

$$h^e_{d,\pi,\langle g^*\rangle}(n) = \max \left\{ \begin{array}{l} m^*, \max\{h^e_{d0^k1,\pi,\emptyset}(n) : |d| + k < \max_{a<c}\{|d_a|\}\}, \\ \min\{k : \forall a < c \forall i \in (d^a)^{-1}(\{1\}) \sigma \in \mathrm{dom}(\pi^a_i)(\sigma \in k^\omega)\} \end{array} \right\}.$$

Each of these satisfies a different condition. h bigger than m^* ensures that (1) holds, the last line ensures that (1a) holds, and the max satisfies a technical requirement we will need later for the induction step.

Case $l+1$:

Fix $\langle d, \pi, F_s\rangle \leq \langle d_s, \pi_s, F_s\rangle$ and $g_i^* \in \omega^{|d|}$ for $i \in [0, \ldots, l]$ such that for all $f \in F_s$, and all $i < l+1$, $f \upharpoonright |d| \neq g_i^*$ and $g_i^* \neq g_j^*$ if $i \neq j$. Then, by our inductive hypothesis, we have that for all $A \subset \overline{g}^*$ with $|A| \leq l$, either case (1) and (1a) hold or case (2) holds. If for any such subset, we see that (2) holds, then by definition, (2) holds of $\overline{f}^*$, and we are done. Otherwise, we will define a function $h^e_{d,\pi,\langle g_i^* : i < l+1\rangle}$ computable from $\bigoplus F_s$ with the desired properties.

Now, define

$$B_{l+1}(j) = \begin{cases} 0 & j < |d| \\ \max\left\{ h^e_{d0^{j-|d|}1,\pi,\langle f_i^*: i<k\rangle}(j) : \begin{array}{l} |\{f_i^*\restriction|d| : i < k\}| = k < l+1, \\ (\forall i) f_i^*\restriction|d| \in \overline{g}^*, \\ f_i^*(t) < B_{l+1}(t) \text{ for } t \geq |d| \end{array} \right\} & |d| \leq j. \end{cases}$$

In order for our proof to work, following the idea of case $l = 1$, we will define a compact space in $(\omega^\omega)^{l+1}$ such that each coordinate is bounded by B_{l+1}. Restricting to the functions in this compact space is sufficient, given that for all $G \subseteq \omega^\omega$ with $G = \{g_i : i < l+1\}$, $g_i\restriction|d| = g_i^*$, if there is $g \in G$ and $j \in \omega$ with $g(j) > B_{l+1}(j)$, then we can find an extension that will make a small convergence.

This is, indeed, true. Assume that there is a function in G exceeding B_{l+1}. Assume that $g(j) > B_{l+1}(j)$ and that, for all $i < l+1$, $m < j$, $g_i(m) \leq B_{l+1}(m)$ (so, $g(j)$ is the first time we are above B_{l+1}). Let $G = G_0 \cup G_1$ such that for all $f \in G_0$, $f(j) > B_{l+1}(j)$ and for all $a \in G_1$, $a(j) \leq B_{l+1}(j)$. Since $|G_1| < l+1$, and we know that for all $f \in G_0$, $f(j) > B_{l+1}(j)$ and so by definition of B_{l+1},

$$f(j) > B_{l+1}(j) \geq h^e_{d0^{j-|d|}1,\pi,\langle a\restriction j+1: a\in G_1\rangle}(j).$$

By our inductive hypothesis (specifically, by (1a)) we have that for all $f \in G_0$, $\langle d0^{j-|d|}1, \pi, F_s \cup \{f\}\rangle$ is compatible with an extension $r \leq \langle d0^{j-|d|}1, \pi, F_s \cup G_1\rangle$ with $\varphi^{\langle d^r,\pi^r\rangle}(n) \downarrow< h^e_{d0^{j-|d|}1,\pi,\langle h\restriction j+1: h\in G_1\rangle}(n)$, and r does not depend on f. This means that $\langle d0^{j-|d|}1, \pi, F_s \cup G_0\rangle$ is compatible with that $r \leq \langle d0^{j-|d|}1, \pi, F_s \cup G_1\rangle$. Notice that, in this case, we have $\langle d0^{j-|d|}1, \pi, F_s \cup G_1\rangle \leq \langle d, \pi, F_s \cup G_1\rangle$ and $\langle d0^{j-|d|}1, \pi, F_s \cup G_0\rangle \leq \langle d, \pi, F_s \cup G_0\rangle$ which means that $\langle d, \pi, F_s \cup G_1 \cup G_0\rangle = \langle d, \pi, F_s \cup G\rangle$ is compatible with that r. In order to make everything work we just need to make sure that $h^e_{d,\pi,\langle g^*_i : i<l+1\rangle}(t) \geq h^e_{d0^{j-|d|}1,\pi,\langle h\restriction j+1 : h \in G_1\rangle}(t)$ for all $t \geq j$ (notice that, to do it, we just need to ask for $h^e_{d,\pi,\langle g^*_i : i<l+1\rangle}$ to be bigger than B_{l+1}. This was the technical requirement necessary in our previous step.)

Now that we know that our function B_{l+1} works as we want. We will create the compact space.

Since B_{l+1} is computable from $\bigoplus F_s$ we have that the space of collections of functions agreeing with g^*_i up to $|d|$ and bounded by B_{l+1} thereafter, defined by

$$C_{l+1} = \{\langle f_i : i < l+1\rangle : f_i \in \omega^\omega, f_i\restriction|d| = g^*_i \ \&\forall j \geq |d| \ f_i(j) \leq B_{l+1}(j)\}$$

is effectivly compact with respect to $\bigoplus F_s$.

Furthermore, fixing n, we define the sets

$$U^m_n = \{\langle h_i : i < l+1\rangle \in C_{l+1} \ : \ \exists q \leq \langle d, \pi, F_s \cup \{h_i : i < l+1\}\rangle \ \varphi_e^{\langle d_q, \pi_q\rangle}(n) \downarrow < m\}.$$

We can do the same as the case $l = 1$. If the compact space is not the union of U^n_m then we can satisfy (2). Otherwise, we can satisfy (1) as we did in $l = 1$. To satisfy (1a), we do the same as in $l = 1$ and we add that $h^e_{d,\pi,\langle g^*_i : i<l+1\rangle}(t) \geq B_{l+1}(t)$ for all $t \geq |d|$. $\square$

2.2.3 Evasion Degrees

Now we will look at the results relating evasion degrees to the the rest of the nodes in the computable version of Cichoń's diagram.

Theorem 2.17. *If A computes a weakly 1-generic, then A is an evasion degree.*

Proof. If A computes a weakly 1-generic, then it computes a function escaping all computably meager sets. Furthermore, the collection of sets predicted by any computable predictor is a computably meager set by Lemma 2.12, and so A computes a function evading any computable predictor. □

Theorem 2.18. *If A is DNC, then A is an evasion degree.*

Proof. Let $\{P_e = \langle D_e, \pi_e \rangle\}$ be a list of the partial computable predictors by index e. We note that by a result of Jockusch in [14], A computes a DNC function if and only if it computes a strongly DNC function—that is, a function $f \leq_T A$ such that for all n, and $\forall e \leq n \; f(n) \neq \varphi_e(e)$. Then we can define $g(m) = f(n_m)$ for n_m large enough that $f(n_m) \neq \pi_e(g\restriction_m)$ for all $e \leq m$. We can effectively find n_m large enough by a simple coding argument. □

Corollary 2.19. *If A is weakly meager engulfing, then A is an evasion degree.*

Proof. By a result of Rupprecht in [30] A is weakly meager engulfing if and only if it is high or DNC. If A is high, then it has hyperimmune degree, and so is an evasion degree by Theorem 2.17 and the fact that hyperimmune degrees compute weakly 1-generics. If A is DNC, then it is an evasion degree by Theorem 2.18. This completes the proof. □

Surprisingly, we actually get an even stronger result, which differs greatly from the analogous case on the set theoretic side:

Corollary 2.20. *If A is not low for weak 1-generics, then A is an evasion degree.*

Proof. By a result of Stephan and Yu in [32], A is not low for weak 1-generics if and only if A is hyperimmune or DNC. Combining this with Theorem 2.17 and Theorem 2.18, we have the desired result. □

Definition 2.21. We define a trace to be a function $g : \omega \to [\omega]^{<\omega}$ with $|g(n)| = n$. A computable trace will simply have g computable.

We define $A \in 2^\omega$ to be computably traceable if for all $f \in \omega^\omega$ with $f \leq_T A$, there is a computable trace g such that $f(n) \in g(n)$ for all n.

Theorem 2.22. *If A is an evasion degree then A is not low for Schnorr tests.*

Proof. Let A be low for Schnorr tests. Then, by a result of Terwijn and Zambella in [34], it follows that A is computably traceable. Let $f \leq_T A$ be a total function. Then we define g by $g(n) = f\restriction_{I_n}$ where $I_n = \left[\frac{n(n-1)}{2}, \frac{n(n+1)}{2}\right)$. (Any computable partition of ω into disjoint sets with $|I_n| = n$ works here.) Note that since $g \leq_T f \leq_T A$, it follows that g is computably traceable. Then, by assumption, there is a computable trace T where $T(n) \subset \omega^n$, $|T(n)| = n$, and $g\restriction_{I_n} \in T(n)$. However, for any n, there are at most $n-1$ values on which a first difference between members of $T(n)$ is witnessed. Put another way, there are at most $n-1$-many values i such that there are $\sigma, \tau \in T(n)$ with $\sigma\restriction_i = \tau\restriction_i$, but $\sigma(i) \neq \tau(i)$. So there must be $j \in I_n$ where for all $\sigma, \tau \in T(n), \sigma\restriction_j = \tau\restriction_j \Rightarrow \sigma(j) = \tau(j)$. Then, we can computably build a predictor which predicts f by adding j to D, and accurately predicting all the elements of the trace. □

To prove the next theorem we will use the notion of clumpy trees introduced by Downey and Greenberg in [8]. A necessary lemma and definitions are reproduced here. K will be used to refer to prefix-free Kolmogorov complexity.

Lemma 2.23. *There is a computable mapping* $(\sigma, \varepsilon) \mapsto n_\varepsilon(\sigma)$ *which maps a finite binary string* $\sigma \in 2^{<\omega}$ *and a rational* $\varepsilon > 0$ *to a natural number* n *such that there is some binary string* τ *of length* n *such that*

$$\frac{K(\sigma\tau)}{|\sigma\tau|} \geq 1 - \varepsilon.$$

Definition 2.24. A *perfect function tree* is a function $T : 2^{<\omega} \to 2^{<\omega}$ that preserves extension and compatibility.

Let T be a perfect function tree, $\sigma \in \operatorname{im} T$, the image of T, and let ε be a positive rational. We say that T *contains an* ε*-clump above* σ if for all binary strings τ of length $n_\varepsilon(\sigma), \sigma\tau = T(\rho\tau)$, where $\sigma = T(\rho)$. We further define T to be ε*-clumpy* if for all $\sigma \in T$, T contains an ε-clump above σ.

Definition 2.25 (Athreya, et al.[1])**.** Given $A \in 2^\omega$, the *effective packing dimension* of A is given by

$$\limsup_{n\to\infty} \frac{K(A\restriction_n)}{n}$$

Theorem 2.26. *There is an* $A \in 2^\omega$ *which is not an evasion degree, but has positive packing dimension.*

Proof. The idea of this proof will be to use forcing with computable trees with some specific properties. First, at the eth stage, we will be pruning to a tree consisting entirely of paths A for which φ_e^A is computably predictable. We will use this to ensure that the result of our forcing does not compute an evading function. Second, the trees will be clumpy, allowing us to choose extensions which occasionally have high relative complexity. This will mean our resulting set has positive packing dimension.

Given an initial segment A_{e-1} and a computable tree T_{e-1} extending this initial segment, we will prune our tree to T_e, so that there is a single predictor that always predicts $\varphi_e^A(n)$ for every remaining path $A \in T_e$ while maintaining the clumpiness requirement.

At every stage in our construction, we will assume that there is no initial segment σ in our current tree T_{e-1} such that φ_e^A is non-total for all paths $A \succ \sigma$. Additionally, we will assume that for any $\sigma \in T_e$, there exist $\tau_1, \tau_2 \succ \sigma$ such that $\varphi_e^{\tau_1} \neq \varphi_e^{\tau_2}$. If either of these fail, we define $A_e = \sigma$ and T_e is the portion of T_{e-1} extending σ. In either case, the clumpiness condition is preserved for the next stage. In the case that the first assumption fails, φ_e^A is not total for all $A \succ \sigma$, and so we need not predict it accurately. In the case that the latter assumption fails, φ_e^A is computable for all $A \succ \sigma$, and so can be predicted easily.

Each run of the construction will go as follows: We will rotate through 3 distinct goals. We can think of them as clumping, differentiating, and predicting.

First, we will add clumps. Given a collection $\{\sigma_i\}$ of initial segments in the tree, each of length n, we will search for $m > n$ such that $T_{e-1}\restriction_m$ contains a 1/2-clump above σ_i for each σ_i. Then, the collection given by $T_{e-1}\restriction_m$ will be the $\{\tau_i\}$ for the next stage.

Next, we will differentiate. We look for $j > m$ so that each m-length τ_i has an extension γ_i of length j such that $\varphi_e^{\gamma_i}$ is distinct for each such γ_i. We are guaranteed to find these by our previous assumption about splitting.

In the final step, we predict. We now look for $d \in \omega$ such that $\varphi_e^{\gamma_i}(d)$ is undefined for all γ_i previously defined. We add this d to D for the predictor we are building, and for each γ_i we look for a further extension $\sigma_i \succ \gamma_i$ such that $\varphi_e^{\sigma_i}(k)\downarrow$ for all $k \leq d$. Then we define $\pi(\varphi_e^{\sigma_i}\restriction_d) = \varphi_e^{\sigma_i}(d)$. For all other strings a of length d, we can define $\pi(a) = 0$. Now, finally, these σ_i become the initial segments of the tree that we start with for the next pass through these three steps. We repeat the process indefinitely.

Finally, once T_e is defined, we will pick $A_e \succ A_{e-1}$ with $|A_e| > 2|A_{e-1}|$ and $\frac{K(A_e)}{|A_e|} > \frac{1}{2}$. Such a string is guaranteed to exist because of the clumpiness condition on our tree.

Then, $A = \bigcup A_e$ is the desired degree, as it is a path through each T_e, and so φ_e^A is computably predictable, but by construction, A has packing dimension $\geq 1/2$. □

Note that there is nothing special about 1/2 in our construction, and a small alteration in the proof can give us A with effective packing dimension of 1.

Lemma 2.27 (Downey and Greenberg[8]). *$A \in 2^\omega$ computably traceable $\Rightarrow$ A has effective packing dimension 0.*

Indeed, this is true of c.e. traceable sets as well.

Corollary 2.28. *There is a degree which is not computably traceable, but not an evasion degree.*

Proof. This is an immediate result of Lemma 2.27 and Theorem 2.26. □

In our finished diagram including prediction and evasion (Figure 4), we have included some of the alternate characterizations of nodes we used that include properties of and relations to the computable functions.

2.3 Rearrangement

The rearrangement number was recently introduced in [4] by Blass, Brendle, Brian, Hamkins, Hardy, and Larson. All results and definitions about this characteristic can be found there.

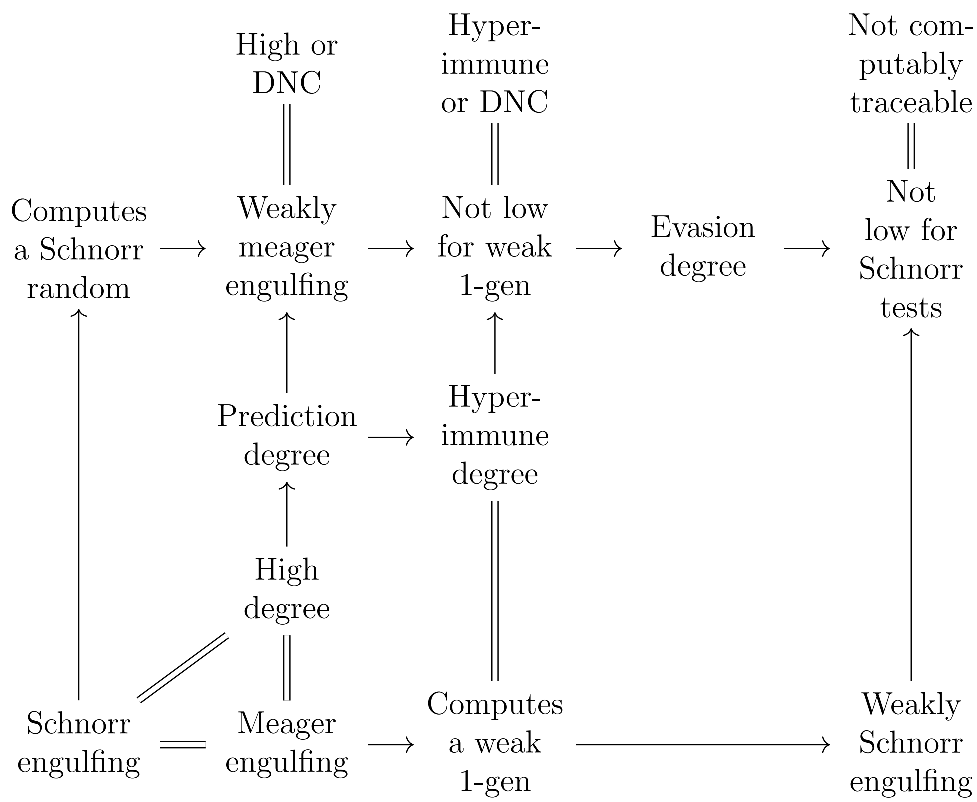

Figure 4: Effective Cichoń's diagram including prediction and evasion degrees.

2.3.1 Definitions

Definition 2.29. The *rearrangement number* $\mathfrak{rr}$ is defined as the smallest cardinality of any family C of permutations of ω such that, for every conditionally convergent series $\sum a_n$ of real numbers, there is a permutation $p \in C$ for which

$$\sum a_{p(n)} \neq \sum a_n.$$

A priori, there are a few different ways of making this happen, namely making the permuted series diverge to infinity, making the permuted series oscillate, and making the permuted series sum to a different finite sum than the original series. In practice, oscillation is easier to achieve than the other two, and so it only makes sense to isolate the other two possibilities, giving a few additional characteristics, where the variation requirement is stronger.

Definition 2.30. We present three additional refinements, giving slightly different characterizations:

- $\mathfrak{rr}_f$ is defined the same way as $\mathfrak{rr}$, but where the sum is required to converge to a different finite number.
- $\mathfrak{rr}_i$ is defined the same way, but the sum is required to diverge to infinity.
- $\mathfrak{rr}_{fi}$ is defined the same way, but the sum is required to either diverge to infinity or converge to a different finite number.

Simply by definition, one can easily see that $\mathfrak{rr} \leq \mathfrak{rr}_{fi} \leq \mathfrak{rr}_f, \mathfrak{rr}_i$. The authors in [4] were able to show that it is consistent that $\mathfrak{rr} < \mathfrak{rr}_{fi}$, but were unable to conclusively show

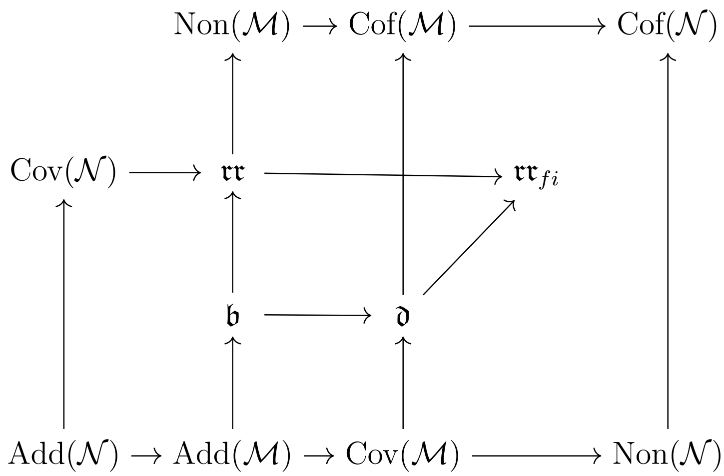

Figure 5: Cichoń's diagram including $\mathfrak{rr}$ and $\mathfrak{rr}_{fi}$.

whether or not the latter three characteristics were separable from each other. Similarly, on the effective side, we have been unable to separate the finite case, the infinite case, or the case allowing either from each other, and so here we will only present the highness notions analogous to $\mathfrak{rr}$ and $\mathfrak{rr}_{fi}$ (although it should be clear what the other two would look like.)

Definition 2.31. We define a conditionally convergent series of rationals $\sum a_n$ to be *computably imperturbable* if, for all computable permutations p, we have that

$$\sum a_n = \sum a_{p(n)}$$

Also, we define $\sum a_n$ to be *weakly computably imperturbable* if no computable permutation p has that either

$$\sum a_{p(n)} = B \neq A = \sum a_n \quad \text{or} \quad \sum a_{p(n)} = \pm\infty.$$

Equivalently, we can define a series to be weakly computably imperturbable if the only way we get inequality of series under computable permutation is by oscillation, that is

$$\sum a_n \neq \sum a_{p(n)} \Rightarrow \sum a_{p(n)} \text{ fails to converges by oscillation.}$$

Finally, we define a real $X \in 2^\omega$ as (weakly) computably imperturbable if it computes a series with the corresponding property.

We present here known facts about $\mathfrak{rr}$ and $\mathfrak{rr}_{fi}$ along with their computable analogs. All results can be found in [4].

Theorem 2.32. *The following relationships are known for* $\mathfrak{rr}$ *and* $\mathfrak{rr}_{fi}$*.*

Cardinal Char.	*Highness Properties*	*Theorem*
$\mathfrak{b} \leq \mathfrak{rr}$	*high* $\Rightarrow$ *imperturbable*	*2.33*
$\mathfrak{d} \leq \mathfrak{rr}_{fi}$	*weak 1-gen* $\Rightarrow$ *weakly imperturbable*	*2.34*
$non(\mathcal{N}) \leq \mathfrak{rr}$	*computes a Schnorr random* $\Rightarrow$ *imperturbable*	*2.43*
$\mathfrak{rr} \leq cov(\mathcal{M})$	*impertubable* $\Rightarrow$ *weakly meager engulfing*	*2.44*
$CON(non(\mathcal{N}) < \mathfrak{rr})$	*imperturbable* $\not\Rightarrow$ *computes a Schnorr random*	*Open*
$CON(\mathfrak{b} < \mathfrak{rr})$	*imperturbable* $\not\Rightarrow$ *high*	*2.45*
$CON(\mathfrak{rr} < \mathfrak{rr}_{fi})$	*weakly imperturbable* $\not\Rightarrow$ *imperturbable*	*2.46*
$CON(\mathfrak{d} < \mathfrak{rr}_{fi})$	*weakly imperturbable* $\not\Rightarrow$ *hyperimmune*	*2.47*

2.3.2 Imperturbability results

The following is an adaptation of Theorems 15 and 16 in [4].

Theorem 2.33. *If* X *is high, then it is imperturbable.*

Proof. Let $X \in 2^\omega$ be high and $\sum a_n$ be any computable conditionally convergent series. By a classic result of Martin in [23], this means that there is a (strictly increasing) function $f \leq_T X$ such that f dominates all computable functions. Let $\sum a_n$ be any computable conditionally convergent series. Define the sequence $\{b_k\}$ by

$$b_k = \begin{cases} a_n & k = f^n(0) \\ 0 & \text{otherwise} \end{cases},$$

using the convention that f^n is the n-times application of f, that is

$$f^n(a) = \overbrace{f(\cdots f(f}^{n}(a))).$$

We claim that $\sum b_{p(n)} = \sum a_n$ for all computable permutations p. To see that this is true, for each $e \in \omega$, we will define a computable function g_e such that if φ_e is a permutation, it follows that $\varphi_e(i) \leq n, g_e(n) \leq \varphi_e(j) \Rightarrow i \leq j$ for all $i, j \in \omega$. Clearly, given such computable functions, we can see that the series $\sum b_k$ defined above has the desired property, as f dominates all of the g_e, and so no computable permutation alters the order of any more than finitely many non-zero elements, leaving the sum unchanged.

In order to define $g_e(n)$, we first assume φ_e is a permutation, if it isn't, nothing that we do matters, as we do not have to defeat it. We begin searching computably for $A_n = \{l \in \omega : \varphi_e(l) \leq n\}$. At some finite stage in our computation, we will have found l_k such that $\varphi_e(l_k) = k$ for all $k \leq n$. This follows from the fact that φ_e is a permutation. Then, let $a = \max\{l_k : k \leq n\}$. Finally, we can define $g_e(n) = \max\{\varphi_e(m) : m \leq a\}$. This g_e has the desired property by construction. $\square$

The following is an adaptation of Theorem 18 in [4].

Theorem 2.34. *If X is of hyperimmune degree, then X is weakly imperturbable.*

Proof. This proof will be very similar to that of Theorem 2.33. Here, let X be of hyperimmune degree. Then, in particular, there is some $f \leq_T X$ such that $f > \varphi_e$ infinitely often for any e. That is, for every e, there are infinitely many n with $f(n) > \varphi_e(n)$. Here, we will also require that f is strictly increasing. Again, for $\sum a_n$ some computable conditionally convergent series, we define the sequence $\{b_k\}$ by

$$b_k = \begin{cases} a_n & k = f^n(0) \\ 0 & \text{otherwise} \end{cases}.$$

We claim that for all $\varepsilon > 0$ and $e \in \omega$, if φ_e is a permutation, then there are infinitely-many distinct pairs $i, j \in \omega$ such that

$$\left| \sum_{n=0}^{i} b_{\varphi_e(n)} - \sum_{n=0}^{j} a_n \right| < \varepsilon.$$

To see that this is true, we can use exactly the same g_e as we used in Theorem 2.33. Remember, if φ_e is a computable permutation, then g_e is total computable. Since f is not dominated by any computable function, it follows that $f(n) > g_e(n)$ infinitely often. In particular, since f is monotone increasing, there must be infinitely-many n so that $f^{n+2}(0) \geq g_e(f^n(0))$. For each such n, there is an initial partial sum of the $b_{\varphi_e(k)}$ which differs from $\sum_{n=0}^{j} a_n$ by at most $|a_{j+1}|$. These pairs have the desired property. Then, since $|a_n| \to 0$ for n large, the initial partial sums of the $b_{\varphi_e(k)}$ are infinitely often arbitrarily close to those of the a_n. It follows that $\sum b_{\varphi_e(k)}$ can neither converge to a different limit than $\sum a_n$, nor diverge to infinity. Thus we have that $\sum b_k$ is a weakly imperturbable sum, as desired. □

For the next lemma we will need the following definitions and facts from [31]:

Definition 2.35. A *computable metric space* is a triple $\mathbb{X} = (X, d, S)$ such that

(1) X is a complete metric space with metric $d : X \times X \to [0, \infty)$.

(2) $S = \{a_i\}_{i \in \omega}$ is a countable dense subset of X.

(3) The distance $d(a_i, a_j)$ is computable uniformly from i and j.

A point $x \in X$ is said to be computable if there is a computable function $h : \omega \to \omega$ such that for all $m > n$, we have $d(a_{h(m)}, a_{h(n)}) \leq 2^{-n}$ and $x = \lim_{n\to\infty} a_{h(n)}$ The sequence $(a_{h(m)})$ is the *Cauchy-name for* x.

Definition 2.36. Let $Y = (\mathbb{Y}, S, d_{\mathbb{Y}})$ be a computable metric space. The space of measurable functions from $(2^\omega, \lambda)$ to $\mathbb{Y}$ is a computable metric space under the metric

$$d_{\text{meas}}(f, g) = \int \min(d_{\mathbb{Y}}, 1)\ d\lambda$$

and test functions of the form $\varphi(x) = c_i 1_{[\sigma_i]}$ when $x \in [\sigma_i]$ (prefix-free $\sigma_0, \ldots, \sigma_{k-1} \in 2^{<\omega}; c_0, \ldots, c_{k-1} \in S$). The computable points in this space are called effectively measurable functions.

Lemma 2.37 (Rute[31])**.** *Suppose $f : (\mathbb{X}, \mu) \to \mathbb{Y}$ is effectively measurable with Cauchy-name (φ_n) in d_{meas}. The limit $\lim_{n\to\infty} \varphi_n(x)$ exists on all Schnorr randoms x.*

Lemma 2.38 (Kolmogorov[18])**.** *Let $X_0, \ldots, X_n$ be independent random variables with expected value $E[X_i] = 0$ and finite variance. Then for each $\lambda > 0$*

$$P\left[\max_{0\leq k\leq n}\left(\sum_{i=0}^{k} X_i\right) \geq \lambda\right] \leq \frac{1}{\lambda^2}\sum_{i=0}^{n} Var(X_i).$$

This collection of lemmas will be used to prove the following result which is an effectivization of a theorem of Rademacher [27].

Lemma 2.39. *If the sequence of rationals $\{a_n\}$ is computable, $\sum a_n^2 < \infty$ is computable, and $X \in 2^\omega$ is a Schnorr random, then $\sum a_n(-1)^{X(n)}$ converges.*

Proof. To see this, we will find a Cauchy-name for the function $f(x) = \sum a_n(-1)^{x(n)}$ in the metric d_{meas}. Then we need only apply Lemma 2.37 to get the desired result.

Given a computable sequence of rationals $\{a_n\}$ with $\sum a_n^2 < \infty$ computable, and $m \in \omega$ we define $\varphi_m(x) = \sum_{n=0}^{i_m} a_n(-1)^{x(n)}$ where i_m is least such that

$$\sum_{n=i_m}^{\infty} a_n^2 < \frac{1}{8^{m+1}}.$$

To see that this is a Cauchy-name, given $j > m$, we have that

$$d_{\text{meas}}(\varphi_j, \varphi_m) \leq \frac{1}{2^{m+1}} + \mu\left\{x \in 2^\omega : \left|\sum_{n=i_m+1}^{i_j} a_n(-1)^{x(n)}\right| > \frac{1}{2^{m+1}}\right\}.$$

However, we can effectively bound the measure of the set in this inequality by

$$\left\{x \in 2^\omega : \left|\sum_{n=i_m+1}^{i_j} a_n(-1)^{x(n)}\right| > \frac{1}{2^{m+1}}\right\} \subset \bigcup_{k=0}^{\infty}\left\{x \in 2^\omega : \left|\sum_{j=i_m}^{i_m+k} a_j(-1)^{x(j)}\right| > \frac{1}{2^{m+1}}\right\}$$

Then, applying Lemma 2.38, we have

$$\bigcup_{k=0}^{\infty}\left\{x \in 2^\omega : \left|\sum_{j=i_m}^{i_m+k} a_j(-1)^{x(j)}\right| > \frac{1}{2^{m+1}}\right\} \leq \frac{1}{(1/2^{m+1})^2}\sum_{j=i_m}^{\infty} a_j^2$$
$$< \frac{1}{2^{m+1}},$$

and so $d_{\text{meas}}(\varphi_m, \varphi_j) \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} = \frac{1}{2^m}$, as desired. Thus, φ_m is a Cauchy name, as desired. Then, by Lemma 2.37, it must converge on all Schnorr randoms. □

Lemma 2.40 (Folklore)**.** *A computable permutation of a Schnorr Random is Schnorr Random.*

The following is an adaptation of Theorem 11 in [4].

Lemma 2.41. *Given a computable permutation p, there is a computable permutation q with the property that there are infinitely many i such that $\{q(n) : n \leq i\} = \{p(n) : n \leq i\}$ and infinitely many j such that $\{q(n) : n \leq j\} = \{0, \ldots, j\}$.*

Proof. We can essentially just build this. Let p be a computable permutation, then we alternate between conditions. We define $q_0(0) = 0$, and then we build q in stages such that the domain of q_s will always be an initial segment of ω. For each $s > 0$, we do the following:

If s is odd, we aim to add an i so that $\{q(n) : n \leq i\} = \{p(n) : n \leq i\}$. To do this, we begin to search computably for $m_k \in \omega$ for k on which q_{s-1} has already been defined such that $p(m_k) = q_{s-1}(k)$ for each $k \in \text{dom}(q_{s-1})$. Then we will define q_s up to $\max\{m_k\}$ by simply building a bijection between $p\{0, \ldots, \max\{m_k\}\}$ picking one element at a time. This is simple, as the collection is computable, and q_{s-1} is already a bijection with a subset, and so we can simply extend. Then, $\max\{m_k\}$ will be the desired i.

If s is even, we aim to add a j so that $\{q(n) : n \leq j\} = \{0, \ldots, j\}$. This is even more straightforward. The j we choose will be $j = \max(\text{range}(q_{s-1}))$, and we can simply build a bijection between the finite, computable, same-size sets, $\{0, \ldots, j\} \setminus \text{range}(q_{s-1})$ and $\{0, \ldots, j\} \setminus \text{dom}(q_{s-1})$ in order to extend q_{s-1} to q_s.

It is straightforward to see that, from the way we constructed q, $q = \bigcup q_s$ is a bijection, and $\text{range}(q) = \text{dom}(q) = \omega$. Thus, q is a computable permutation, and has the desired property. □

Note, this result can actually be extended so that, given any two permutations p_1, p_2, there is a permutation $q \leq_T p_1 \oplus p_2$ such that there are infinitely many i, j such that $\{q(n) : n \leq i\} = \{p_1(n) : n \leq i\}$ and $\{q(n) : n \leq i\} = \{p_2(n) : n \leq i\}$.

The following is an adaptation of Theorem 6 in [4].

Lemma 2.42. *If $\sum a_n$ is not computably imperturbable, then there is a computable permutation p such that $\sum a_{p(n)}$ fails to converge due to oscillation.*

Proof. Let $\sum a_n$ be a series which is not computably imperturbable. That is, there is a computable permutation p such that

$$\sum a_n \neq \sum a_{p(n)}.$$

We can assume that $\sum a_{p(n)} = \pm\infty$ or $\sum a_{p(n)} = B \neq A = \sum a_n$, otherwise there is nothing to show. Now let q be as in Lemma 2.41. This q has the desired property. If $\sum a_{p(n)} = \infty$, then for i as in the lemma, we have that

$$\sum_{n=0}^{i} a_{q(n)} = \sum_{n=0}^{i} a_{p(n)},$$

thus we can see that these partial sums grow without bound, but simultaneously, for j as in the lemma, we have that

$$\sum_{n=0}^{j} a_{q(n)} = \sum_{n=0}^{j} a_n,$$

and so these partial sums tend towards $A = \sum a_n$. Thus, the whole series must be non-convergent due to oscillation. A similar argument shows that if $\sum a_{p(n)} = B \neq A$, then there are infinite subsequences of initial sums of $\sum a_{q(n)}$ converging to both A and B, which also means that $\sum a_{q(n)}$ must be non-convergent due to oscillation. □

Theorem 2.43. *If X computes a Schnorr Random, then X is imperturbable.*

Proof. Let $X \in 2^\omega$ and $A \leq_T X$ be Schnorr Random. Then, we claim that if we define $a_n = \frac{(-1)^{A(n)}}{n}$, the series $\sum a_n$ is imperturbable. To see this, let p be a computable permutation, then $\sum a_{p(n)}$ converges by Lemma 2.39 and Lemma 2.40. Namely, the sequence $\left\{\frac{1}{p(n)}\right\}$ is a computable sequence by construction,

$$\sum \left(\frac{1}{p(n)}\right)^2 = \sum \frac{1}{n^2} = \frac{\pi^2}{6}$$

is computably converging to a computable sum, and the indices of negative entries of our sequence is Schnorr Random by Lemma 2.40. Thus, we can apply Lemma 2.39, and so the series converges for all computable permutations. Further, since this series must converge for all computable permutations, it follows from Lemma 2.42 that it must be imperturbable. □

Theorem 2.44. *If X is imperturbable, then X is weakly meager engulfing.*

Proof. We will actually show that X is weakly meager engulfing in the space of permutations, but there is a computable bijection between Let X imperturbable, then there is a conditionally convergent imperturbable series $\sum a_n \leq_T X$. We claim that the set of permutations leaving this sum unchanged is contained in an X-effectively meager set. In particular, the set of permutations which do not make the sum $+\infty$ is exactly the set

$$E = \bigcup_{k \in \omega} \bigcap_{m \geq k} \left\{ p : \sum_{n=0}^{m} a_{p(n)} \leq k \right\}$$

Now, we simply observe that the intersection

$$E_k = \bigcap_{m \geq k} \left\{ p : \sum_{n=0}^{m} a_{p(n)} \leq k \right\}$$

is Σ^0_1 in X, additionally, it is nowhere dense, as any initial segment which falls in the appropriate range can then have all terms of the same sign for long enough to escape the interval.

Thus, E is an X-effectively meager set of permutations containing all computable permutations, as desired. □

We can immediately see that almost all of the forgoing implications are not reversible. This follows from the theorems plus existing, known cuts of the computable Cichoń's diagram. These cuts are cataloged in [6] §4.2.

Corollary 2.45. *There is an X which is imperturbable but not high.*

Proof. This is a direct result of Theorem 2.43 plus the fact that there is a Schnorr random which is not high. In fact, there is a low ML-random, which we can see from the low basis theorem plus the existence of a universal ML-test. See e.g. [26] Theorem 1.8.37. □

Corollary 2.46. *There is an X which is weakly imperturbable but not imperturbable.*

Proof. We will use the fact that weakly meager engulfing is equivalent to high or DNC, a proof of which can be found in [16]. The corollary follows directly from Theorems 2.34 and 2.44 plus the existence of a set of hyperimmune degree which is not weakly meager engulfing. Any nonrecursive low r.e. set suffices. Obviously, being of hyperimmune degree means that it is also weakly computably imperturbable. Additionally, by Arslanov's completeness criterion ([26], 4.1.11), such a set cannot be DNC, and is not high by definition. Thus, the set is also not weakly meager engulfing. □

Corollary 2.47. *There is an X which is weakly imperturbable and hyperimmune-free.*

Proof. This follows directly from Theorem 2.43 plus the fact that imperturbable implies weakly imperturbable and the existence of a Schnorr random which is hyperimmune-free. The fact follows by taking a set A of hyperimmune-free PA degree (see e.g. [26] 1.8.32 and 1.8.42). □

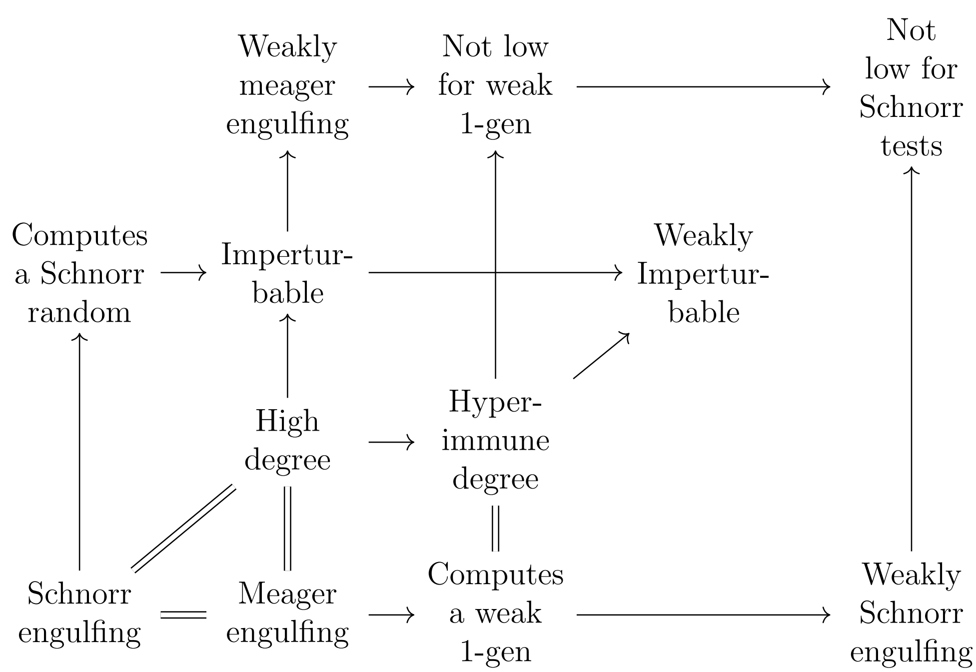

Figure 6: Effective Cichoń's diagram including imperturbability.

Bibliography

[1] K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo, *Effective strong dimension in algorithmic information and computational complexity*, in Annual Symposium on Theoretical Aspects of Computer Science, Springer, 2004, pp. 632–643.

[2] G. Barmpalias, J. S. Miller, and A. Nies, *Randomness notions and partial relativization*, Israel Journal of Mathematics, 191 (2012), pp. 791–816.

[3] A. Blass, *Cardinal characteristics and the product of countably many infinite cyclic groups*, J. Algebra, 169 (1994), pp. 512–540.

[4] A. Blass, J. Brendle, W. Brian, J. D. Hamkins, M. Hardy, and P. B. Larson, *The rearrangement number*, arXiv preprint arXiv:1612.07830, (2016).

[5] J. Brendle, *Evasion and prediction—the Specker phenomenon and Gross spaces*, in Forum Mathematicum, vol. 7, 1995, pp. 513–542.

[6] J. Brendle, A. Brooke-Taylor, K. M. Ng, and A. Nies, *An analogy between cardinal characteristics and highness properties of oracles*, arXiv preprint arXiv:1404.2839, (2014).

[7] J. Brendle and S. Shelah, *Evasion and prediction II*, Journal of the London Mathematical Society, 53 (1996), pp. 19–27.

[8] R. Downey and N. Greenberg, *Turing degrees of reals of positive effective packing dimension*, Information Processing Letters, 108 (2008), pp. 298–303.

[9] R. Downey, W. Merkle, and J. Reimann, *Schnorr dimension*, Mathematical Structures in Computer Science, 16 (2006), pp. 789–811.

[10] R. Downey, A. Nies, R. Weber, and L. Yu, *Lowness and nullsets*, The Journal of Symbolic Logic, 71 (2006), pp. 1044–1052.

[11] I. Herbert, *A perfect set of reals with finite self-information*, The Journal of Symbolic Logic, 78 (2013), pp. 1229–1246.

[12] W. P. Hudelson, *Partial randomness and Kolmogorov complexity*, PhD thesis, The Pennsylvania State University, 2013.

[13] C. G. Jockusch, *Degrees in which the recursive sets are uniformly recursive*, Canadian Journal of Mathematics, 24 (1972), pp. 1092–1099.

[14] ——, *Degrees of functions with no fixed points*, Studies in Logic and the Foundations of Mathematics, 126 (1989), pp. 191–201.

[15] M. Kada, *The Baire category theorem and the evasion number*, Proceedings of the American Mathematical Society, 126 (1998), pp. 3381–3383.

[16] B. Kjos-Hanssen, W. Merkle, and F. Stephan, *Kolmogorov complexity and the recursion theorem*, in Annual Symposium on Theoretical Aspects of Computer Science, Springer, 2006, pp. 149–161.

[17] B. Kjos-Hanssen, J. S. Miller, and R. Solomon, *Lowness notions, measure and domination*, Journal of the London Mathematical Society, (2012).

[18] A. Kolmogoroff, *Über die summen durch den zufall bestimmter unabhängiger grössen*, Mathematische Annalen, 99 (1928), pp. 309–319.

[19] S. Kurtz, *Randomness and genericity in the degrees of unsolvability.*, Dissertation Abstracts International Part B: Science and Engineering[DISS. ABST. INT. PT. B- SCI. & ENG.],, 42 (1982), p. 1982.

[20] S. Lempp, J. S. Miller, K. M. Ng, D. D. Turetsky, and R. Weber, *Lowness for effective Hausdorff dimension*, Journal of Mathematical Logic, 14 (2014).

[21] J. H. Lutz, *Dimension in complexity classes*, in Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity, IEEE Computer Society, 2000, pp. 158–169.

[22] ——, *Gales and the constructive dimension of individual sequences*, in Automata, languages and programming, Springer, 2000, pp. 902–913.

[23] D. A. Martin, *Classes of recursively enumerable sets and degrees of unsolvability*, Mathematical Logic Quarterly, 12 (1966), pp. 295–310.

[24] P. Martin-Löf, *The definition of random sequences*, Information and control, 9 (1966), pp. 602–619.

[25] J. S. Miller, *Extracting information is hard: a Turing degree of non-integral effective Hausdorff dimension*, Advances in Mathematics, 226 (2011), pp. 373–384.

[26] A. Nies, *Computability and randomness*, vol. 51 of Oxford Logic Guides, Oxford University Press, 2009.

[27] H. Rademacher, *Einige sätze über reihen von allgemeinen orthogonalfunktionen*, Mathematische Annalen, 87 (1922), pp. 112–138.

[28] J. Reimann, *Computability and fractal dimension*, PhD thesis, 2004.

[29] J. Reimann and F. Stephan, *On hierarchies of randomness tests*, in Proceedings of the 9th Asian Logic Conference, World Scientific Publishing, World Scientific, 2006, pp. 16–19.

[30] N. A. Rupprecht, *Effective correspondents to cardinal characteristics in Cichoń's diagram*, PhD thesis, The University of Michigan, 2010.

[31] J. Rute, *Algorithmic randomness, martingales, and differentiability I*, preprint, (2012).

[32] F. Stephan and L. Yu, *Lowness for weakly 1-generic and Kurtz-random*, in International Conference on Theory and Applications of Models of Computation, Springer, 2006, pp. 756–764.

[33] S. A. Terwijn, *Complexity and randomness*, tech. rep., Department of Computer Science, The University of Auckland, New Zealand, 2003.

[34] S. A. Terwijn and D. Zambella, *Computational randomness and lowness*, The Journal of Symbolic Logic, 66 (2001), pp. 1199–1205.